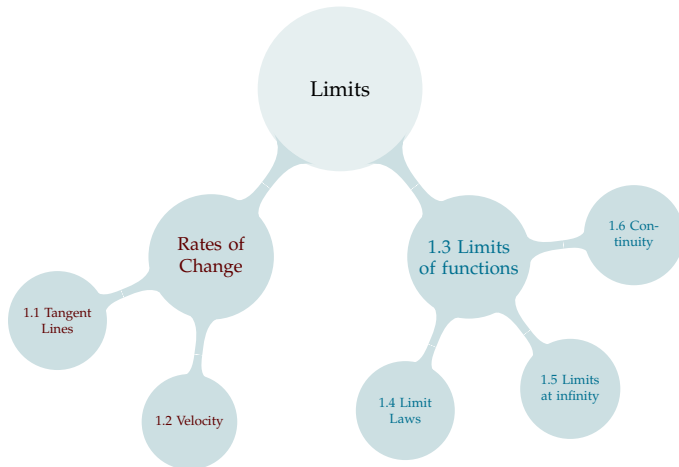
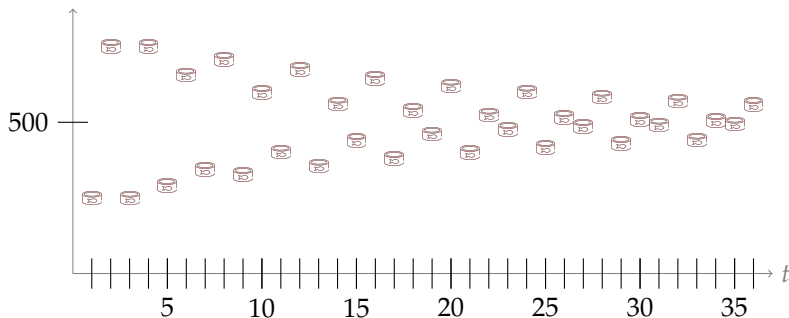


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## 1.8 (Optional) Making Infinite Limits a Little More Formal



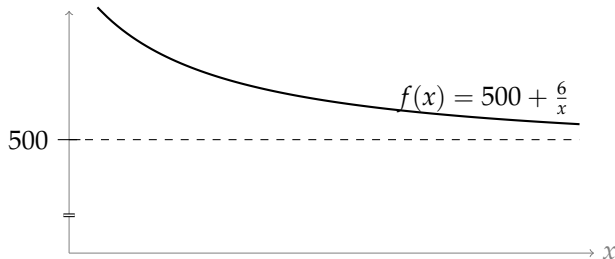
You work at a salmon cannery, putting salmon into cans. Each can is supposed to contain the amount of salmon shown on the label, but some error is allowable. As you work longer, and get more experience, the amount of error you are allowed to have gets smaller.



Was there a time after which your error always *less than* 250 g?

Suppose the amount of salmon that is supposed to be in a can is 500 g. The amount of salmon you put into a can at time  $x$  is  $500 + \frac{6}{x}$ .

You need to reassure your boss that, after some time, your error is never more than 3 g. Find such a time.



No matter how exacting your boss is, if they give you a non-zero error allowance, you can *always* schedule a time after which you will meet their standards.

### Definition 1.8.1 (a)

Let  $f$  be a function defined on the whole real line. We say that

*the limit as  $x$  approaches  $\infty$  of  $f(x)$  is  $L$*

and write

$$\lim_{x \rightarrow \infty} f(x) = L$$

if and only if for every  $\epsilon > 0$  there exists  $M \in \mathbb{R}$  so that  $|f(x) - L| < \epsilon$  whenever  $x > M$ .

Similarly we write

$$\lim_{x \rightarrow -\infty} f(x) = K$$

if and only if for every  $\epsilon > 0$  there exists  $N \in \mathbb{R}$  so that  $|f(x) - K| < \epsilon$  whenever  $x < N$ .

Let  $f$  be a function defined on the whole real line.

We say that “the limit as  $x$  approaches  $\infty$  of  $f(x)$  is  $L$ ” and write

$$\lim_{x \rightarrow \infty} f(x) = L$$

if and only if for every  $\epsilon > 0$

there exists  $M \in \mathbb{R}$  so that  
 $|f(x) - L| < \epsilon$  whenever  $x > M$ .

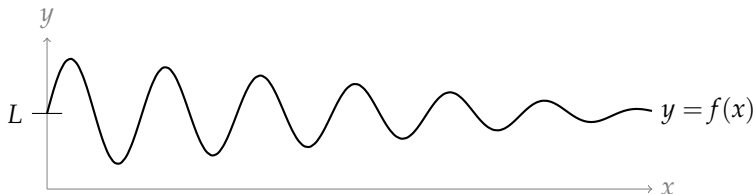
$f(x)$ : actual can weights

$L$ : weight on the label that you want to match

$\epsilon$ : amount of allowable error

$M$ : time after which your weights are always off by less than  $\epsilon$

$|f(x) - L|$ : error (difference between actual amount and label)



Let  $f$  be a function defined on the whole real line. We say that “the limit as  $x$  approaches  $\infty$  of  $f(x)$  is  $L$ ” and write

$$\lim_{x \rightarrow \infty} f(x) = L$$

if and only if for every  $\epsilon > 0$  there exists  $M \in \mathbb{R}$  so that  $|f(x) - L| < \epsilon$  whenever  $x > M$ .

### Definition 1.8.1 (a)

Let  $f$  be a function defined on the whole real line. We say that  $\lim_{x \rightarrow \infty} f(x) = L$  if and only if for every  $\epsilon > 0$  there exists  $M \in \mathbb{R}$  so that  $|f(x) - L| < \epsilon$  whenever  $x > M$ .

Prove, using Definition 1.8.1, that  $\lim_{x \rightarrow \infty} \left[ \frac{2}{x} + 1 \right] = 1$ .

### Definition 1.8.1 (a)

Let  $f$  be a function defined on the whole real line. We say that  $\lim_{x \rightarrow \infty} f(x) = L$  if and only if for every  $\epsilon > 0$  there exists  $M \in \mathbb{R}$  so that  $|f(x) - L| < \epsilon$  whenever  $x > M$ .

Prove, using Definition 1.8.1, that  $\lim_{x \rightarrow \infty} [5e^{-x}] = 0$



## Definition 1.8.1 (a)

Let  $f$  be a function defined on the whole real line. We say that  $\lim_{x \rightarrow \infty} f(x) = L$  if and only if for every  $\epsilon > 0$  there exists  $M \in \mathbb{R}$  so that  $|f(x) - L| < \epsilon$  whenever  $x > M$ .

NOW  
YOU



Prove, using Definition 1.8.1, that  $\lim_{x \rightarrow \infty} \left[ \frac{\sin x}{x} \right] = 0$

## Definition 1.8.1 (a)

Let  $f$  be a function defined on the whole real line. We say that  $\lim_{x \rightarrow \infty} f(x) = L$  if and only if for every  $\epsilon > 0$  there exists  $M \in \mathbb{R}$  so that  $|f(x) - L| < \epsilon$  whenever  $x > M$ .

NOW  
YOU



Prove, using Definition 1.8.1, that  $\lim_{x \rightarrow \infty} \left[ \frac{2x^2}{x^2 + 1} \right] = 2$

### Definition 1.8.1 (a)

Let  $f$  be a function defined on the whole real line. We say that  $\lim_{x \rightarrow \infty} f(x) = L$  if and only if for every  $\epsilon > 0$  there exists  $M \in \mathbb{R}$  so that  $|f(x) - L| < \epsilon$  whenever  $x > M$ .

Prove, using Definition 1.8.1, that  $\lim_{x \rightarrow \infty} 5 = 5$

### Definition 1.8.1 (a)

Let  $f$  be a function defined on the whole real line. We say that  $\lim_{x \rightarrow \infty} f(x) = L$  if and only if for every  $\epsilon > 0$  there exists  $M \in \mathbb{R}$  so that  $|f(x) - L| < \epsilon$  whenever  $x > M$ .

True or False?  $\lim_{x \rightarrow \infty} \sin x = 0$

## USEFUL GENERAL PRINCIPLES

When we showed  $\lim_{x \rightarrow \infty} \left[ \frac{\sin x}{x} \right] = 0$ , we chose  $M$  using:

$$\left| \frac{\sin x}{x} \right| \leq \left| \frac{1}{x} \right| = \frac{1}{x} < \epsilon$$

- ▶  $\left| \frac{1}{x} \right| = \frac{1}{x}$  only when  $x$  is positive. We want to show that an inequality holds for *large enough* values of  $x$ , so if it helps our cause, we can say “make sure  $x$  is larger than *blah*.” Then we just choose  $M$  to be at least that number *blah*.
- ▶ If  $a < b < c$ , then  $a < c$ . So if you want to solve  $a < c$ , but it's too hard to find *exactly* when that's true, see whether you can replace  $a$  with a larger, easier expression  $b$ .

That's what we did when we said  $\left| \frac{\sin x}{x} \right| \leq \left| \frac{1}{x} \right|$ .

# LIMIT AS $x$ GOES TO NEGATIVE INFINITY

## Definition 1.8.1 (a)

We write

$$\lim_{x \rightarrow -\infty} f(x) = K$$

if and only if for every  $\epsilon > 0$  there exists  $N \in \mathbb{R}$  so that  $|f(x) - K| < \epsilon$  whenever  $x < N$ .

Use Definition 1.8.1 to prove  $\lim_{x \rightarrow -\infty} \frac{x^3}{x^3 + 1} = 1$

LIMIT AS  $x$  GOES TO NEGATIVE INFINITY

## Definition 1.8.1 (a)

We write

$$\lim_{x \rightarrow -\infty} f(x) = K$$

if and only if for every  $\epsilon > 0$  there exists  $N \in \mathbb{R}$  so that  $|f(x) - K| < \epsilon$  whenever  $x < N$ .

Use Definition 1.8.1 to prove  $\lim_{x \rightarrow -\infty} \frac{\cos x}{\sqrt{x^4 + x^2 + 1}} = 0$

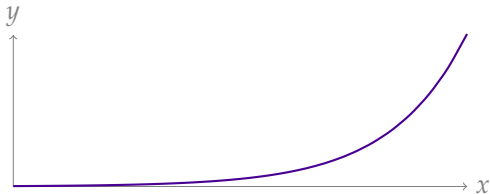
## INFINITE LIMITS

## Definition 1.8.1 (c)

Let  $f$  be a function defined on the whole real line. We write

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

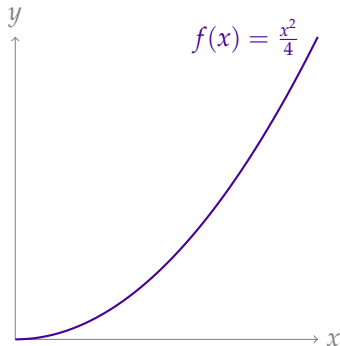
if and only if for every  $P > 0$  there exists  $M > 0$  so that  $f(x) > P$  whenever  $x > M$ .





## Definition 1.8.1 (c)

Let  $f$  be a function defined on the whole real line. We write  $\lim_{x \rightarrow \infty} f(x) = \infty$  if and only if for every  $P > 0$  there exists  $M > 0$  so that  $f(x) > P$  whenever  $x > M$ .



Let

$$P = 1P = 2P = 1\,000\,000P > 0.$$

Find  $M > 0$  so that  $f(x) > P$   
whenever  $x > M$ .

### Definition 1.8.1 (c)

Let  $f$  be a function defined on the whole real line. We write  $\lim_{x \rightarrow \infty} f(x) = \infty$  if and only if for every  $P > 0$  there exists  $M > 0$  so that  $f(x) > P$  whenever  $x > M$ .

Using definition 1.8.1, prove or disprove the following:

$$\lim_{x \rightarrow \infty} \sqrt[3]{x} = \infty$$

### Definition 1.8.1 (c)

Let  $f$  be a function defined on the whole real line. We write  $\lim_{x \rightarrow \infty} f(x) = \infty$  if and only if for every  $P > 0$  there exists  $M > 0$  so that  $f(x) > P$  whenever  $x > M$ .

Using definition 1.8.1, prove or disprove the following:

$$\lim_{x \rightarrow \infty} x(\sin x + 1) = \infty$$

### Definition 1.8.1 (c)

Let  $f$  be a function defined on the whole real line. We write  $\lim_{x \rightarrow \infty} f(x) = \infty$  if and only if for every  $P > 0$  there exists  $M > 0$  so that  $f(x) > P$  whenever  $x > M$ .

Using definition 1.8.1, prove or disprove the following:

$$\lim_{x \rightarrow \infty} x(\sin x + 2) = \infty$$

## Included Work



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