Course Outline 2023 MATH 257/316: Partial Differential Equations

Prerequisites: One of Math 215, 255, 256, 258.

Credits: 3 Credits. Credit only given for one of Math. 257, 316, 358, MECH 358, PHYS 312

Learning Objectives: This course introduces the heat, wave, and Laplace equations in different physical contexts. Students are taught to formulate and implement finite difference numerical solution schemes as well as analytic methods to solve homogeneous boundary value problems (BVP) via separation of variables and Fourier Series and inhomogeneous BVP using eigenfunction expansions.

Instructor: Anthony Peirce, **Office:** Mathematics Building 108, **Home Page:** http://www.math.ubc.ca/˜peirce

Office Hours: Monday: 12:30-1:30 pm, Wed: 3-3:55 pm, Fri: 10-11 am.

Test Dates: Wednesday, October 25th, Wednesday, November 29 th.

Assessment: The final grades will be based on homework (10%) including EXCEL or MATLAB projects, two in-class midterm exams (2x20%=40%), and one final exam (50%). A student must get at least 40% on the final exam to pass this course.

Missing exams and homework deadlines: There are no make-up quizzes or assignments in this course. If you miss any of the assessment deadlines for a valid reason, the weight of that assessment will be transferred to the final exam.

Text: A comprehensive set of lecture notes will be posted online. Any edition of *Elementary Differential Equations & Boundary Value Problem* by W.E. Boyce & R.C. DiPrima, (John Wiley & Sons) will serve as an optional text.

Topics: (Chapters refer to the 2012 Edition of Boyce & DiPrima)	Approximate Time
1. Review of techniques to solve ODEs	1 hr
2. Series Solutions of variable coefficient ODEs (Chapter 5)	
a. Series solutions at ordinary points (5.1-5.3)	3 hrs
b. Regular singular points (5.4-5.7, 5.8 briefly)	4 hrs
3. Introduction to PDEs (Chapter 10): heat equation (10.5), wave equation (10.7), Laplace equation (10.8)	2 hrs
4. Introduction to numerical methods for PDEs using spread sheets	3 hrs
a. First and second derivative approximations using finite differences - errors	
b. Explicit finite difference schemes for the heat equation	
- Stability and derivative boundary conditions	
c. Explicit finite difference schemes for the wave equation	
d. Finite difference approximation of Laplace Equation and iterative methods	
5. Fourier Series and Separation of Variables (Chapter 10)	
a. The heat equation and Fourier Series (10.1-10.6)	9 hrs
b. The wave equation (10.7)	3 hrs
c. Laplace equation (10.8)	5 hrs
6. Boundary Value Problems and Sturm-Liouville Theory (Chapter 11)	
a. Eigenfunctions and eigenvalues (11.1)	1 hr
b. Sturm-Liouville boundary value problems (11.2)	1 hr
c. Nonhomogeneous boundary value problems (11.3)	2 hrs
Tests	2 hrs
Total:	36 hrs

Math 257-316 PDE Formula sheet - final exam

Trigonometric and Hyperbolic Function identities

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$ $\qquad \sin^2 t + \cos^2 t = 1$
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\beta\sin\alpha.$ $\qquad \sin^2 t = \frac{1}{2}(1-\cos(2t))$
$\sinh(\alpha \pm \beta) = \sinh\alpha\cosh\beta \pm \sinh\beta\cosh\alpha$ $\qquad \cosh^2 t - \sinh^2 t = 1$
$\cosh(\alpha \pm \beta) = \cosh\alpha\cosh\beta \pm \sinh\beta\sinh\alpha.$ $\qquad \sinh^2 t = \frac{1}{2}(\cosh(2t)-1)$

Basic linear ODE's with real coefficients

	constant coefficients	**Euler eq**
ODE	$ay''+by'+cy=0$	$ax^2y''+bxy'+cy=0$
indicial eq.	$ar^2+br+c=0$	$ar(r-1)+br+c=0$
$r_1 \neq r_2$ real	$y = Ae^{r_1 x}+Be^{r_2 x}$	$y = Ax^{r_1}+Bx^{r_2}$
$r_1 = r_2 = r$	$y = Ae^{rx}+Bxe^{rx}$	$y = Ax^r + Bx^r\ln\lvert x\rvert$
$r = \lambda \pm i\mu$	$e^{\lambda x}[A\cos(\mu x)+B\sin(\mu x)]$	$x^\lambda[A\cos(\mu\ln\lvert x\rvert)+B\sin(\mu\ln\lvert x\rvert)]$

Series solutions for $y''+p(x)y'+q(x)y=0$ ($\star$) around $x=x_0$.

Ordinary point x_0: Two linearly independent solutions of the form:

$$y(x) = \textstyle\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

Regular singular point x_0: Rearrange ($\star$) as:
$(x-x_0)^2y'' + [(x-x_0)p(x)](x-x_0)y' + [(x-x_0)^2q(x)]y = 0$
If $r_1 > r_2$ are roots of the indicial equation: $r(r-1)+br+c=0$ where $b = \lim_{x\to x_0}(x-x_0)p(x)$ and $c = \lim_{x\to x_0}(x-x_0)^2q(x)$ then a solution of ($\star$) is

$$y_1(x) = \textstyle\sum_{n=0}^{\infty} a_n(x-x_0)^{n+r_1} \quad \text{where } a_0 = 1.$$

The second linerly independent solution y_2 is of the form:
Case 1: If $r_1 - r_2$ is neither 0 nor a positive integer:

$$y_2(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^{n+r_2} \text{ where } \ b_0 = 1.$$

Case 2: If $r_1 - r_2 = 0$:

$$y_2(x) = y_1(x)\ln(x-x_0) + \sum_{n=1}^{\infty} b_n(x-x_0)^{n+r_2} \text{ for some } b_1, b_2\ldots$$

Case 3: If $r_1 - r_2$ is a positive integer:

$$y_2(x) = ay_1(x)\ln(x-x_0) + \sum_{n=0}^{\infty} b_n(x-x_0)^{n+r_2} \text{ where } \ b_0 = 1.$$

Fourier, sine and cosine series

Let $f(x)$ be defined in $[-L,L]$then its Fourier series $Ff(x)$ is a $2L$-periodic function on $\mathbf{R}$:

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos(\frac{n\pi x}{L}) + b_n\sin(\frac{n\pi x}{L})\right\}$$

where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx$
Theorem (Pointwise convergence) If $f(x)$ and $f'(x)$ are piecewise continuous, then $Ff(x)$ converges for every x to $\frac{1}{2}[f(x-)+f(x+)]$.
Parseval's indentity

$$\frac{1}{L}\int_{-L}^{L}|f(x)|^2dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}\left(|a_n|^2+|b_n|^2\right).$$

For $f(x)$ defined in $[0,L]$, its cosine and sine series are

$$Cf(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(\frac{n\pi x}{L}), \quad a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})\,dx,$$

$$Sf(x) = \sum_{n=1}^{\infty} b_n\sin(\frac{n\pi x}{L}), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\,dx.$$

Sturm-Liouville Eigenvalue Problems

ODE: $[p(x)y']' - q(x)y + \lambda r(x)y = 0, \quad a < x < b.$
BC: $\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0.$
Hypothesis: p, p', q, r continuous on $[a,b]$. $p(x) > 0$ and $r(x) > 0$ for $x \in [a,b]$. $\alpha_1^2+\alpha_2^2 > 0$. $\beta_1^2+\beta_2^2 > 0$.
Properties (1) The differential operator $Ly = [p(x)y']' - q(x)y$ is symmetric in the sense that $(f, Lg) = (Lf, g)$ for all f, g satisfying the BC, where $(f,g) = \int_a^b f(x)g(x)\,dx$. (2) All eigenvalues are real and can be ordered as $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$, and each eigenvalue admits a unique (up to a scalar factor) eigenfunction ϕ_n.
(3) **Orthogonality**: $(\phi_m, r\phi_n) = \int_a^b \phi_m(x)\phi_n(x)r(x)\,dx = 0$ if $\lambda_m \neq \lambda_n$.
(4) **Expansion**: If $f(x) : [a,b] \to \mathbf{R}$ is square integrable, then

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \ a < x < b \ , \ c_n = \frac{\int_a^b f(x)\phi_n(x)r(x)\,dx}{\int_a^b \phi_n^2(x)r(x)\,dx}, \ n = 1, 2, \ldots$$