Topics in Geometry: Introduction to Knot Theory

Math 309

This is a math class about knots. While these are objects that arise relatively naturally in a variety of day-to-day and scientific settings, our challenge will be to carefully sort out how to describe, and ultimately study, knots using mathematical tools. The image on the right is taken from Mathematical Research Postcards (Volume 1 Number 1).

Here are some further remarks about this class:

- In a mathematical context, the study of knots falls into a broader area of research called *topology*, which is a branch of geometry (in a broad sense). However, topology (Math 426, for instance) **is not** a prerequisite for this course and as such we won't really take this point of view; our course will be more combinatorial in nature, building from the ground up.

- The course does, on the other hand, have prerequisites in the form of linear algebra (Math 221/223 or similar) and mathematical proof (Math 220 or similar). **We will rely on both.**

- This course will take the point of view that knots are naturally occurring objects–worthy of study in their own right. While I will draw on examples found "in nature" from time to time, this won't be the main emphasis of the course. Instead, this course is meant to provide a window to pure mathematics research, and hopefully provide a partial answer to the question "what do mathematicians do?"

- The bulk of your grade will come from assigned homework (more on this below). Solution sets for these assignments will not be provided. The reason for this is, in part, that there are multiple correct solutions to the majority of the problems set as homework. To succeed in this course it is crucial that you understand *your* solution to a given problem. And to that end, I am happy to discuss your solutions with you during office hours.

Assignment submission and grading will make use of Canvas. Here are some essential guidelines for assignment submission:

1. Your solutions to assignments must be submitted using canvas. In particular, **do not email assignments**.

2. Your solutions must be submitted as a single file.

3. The only accepted file type is PDF.

4. Files should be named according to the following convention: YourName_YourStudentNumber_AssignmentNumber.PDF

There are free online tools available that will allow you to complete (2) and (3). Due to the volume of data being managed, it is really important that these 4 points be met so that we can grade and keep track of your work effectively. As a result, if one (or more) of these four points are not met, your assignment will not be accepted/graded.

To view the full content of this course, including lecture summaries and assignments, turn off reader-view.

Course Units

- **Unit 1: Foundations**
  The relevant sections from Adams are 1.1 through 1.5.
- Lecture 1 (September 5)
  Prelude: parametrizing curves in the torus
  As a point of entry for studying knots, we reviewed some material from calculus about parametrized curves, and looked specifically at some parametrized curves that wind around in a torus.

Lecture 2 (September 10)
Knot diagrams
After reviewing some of the material from last time, we looked carefully at a diagram for the trefoil knot. This led us to a reasonable definition of a knot diagram, which is a choice of planar representative for a knot. We now need to carfuly consider what sort of equivalence there are between diagrams that represent the same knot.

Lecture 3 (September 12)
Reidemeister's theorem
Deciding if two knot diagrams represent the same knot can be hard. Fortunately, there is a theorem, due to Reidemeister, that tells us precisely when this is the case: This gives a (short!) list of possible moves, and guarantees that two diagrams that represent the same knot will always be related by some collection of these moves applied in sequence. The list of required moves may be very long, and it is definitely not unique, so this theorem is hard to apply. But it is useful! We'll see how to use it to construct tools used to distinguish knots.

Lecture 4 (September 17)
Colouring knots
We've now shown that interesting knotting exits. We did this by finding a property of a knot diagram (tricolourability), unchanged under Reidemeister moves, that was enjoyed by the trefoil knot but not by the trivial knot.

- **Assignment 1** (linked here) is due Friday September 20 by 10:59pm.

- **Unit 2: Linking and knotting**
  The relevant sections from Adams are 1.4, 3.1, 3.3, and 5.1.
- Lecture 5 (September 19)
  Invariants of knots and links
  We revisited the work from the last lecture in a more general context: a

knot invariant is a rule F that assigns a value F(K) (a number, a polynomial, a ...) to a given knot K. This should satisfy the property that F(K)=F(K') whenever K and K' are equivalent knots. Invariants also exist to study linking, and we saw that the linking number between components of a link gave an integer-valued invariant.

Lecture 6 (September 24)
Unknotting numbers
We considered two natural but very difficult to compute numerical invariants of knots: the crossing number and the unknotting number. In general, the unknotting number of a knot need not be realize on a minimal diagram, that is, a diagram realizing the crossing number of the knot. By contrast, amazingly, McCoy proved that when K is an alternating knot with unknotting number 1, any alternating diagram for K contains an unknotting crossing.

Lecture 7 (September 26)
Torus knots
A large supply of non-alternating knots are provided by torus knots, which we encountered in earlier lectures. I gave a survey of known results pertaining to crossing number and unknotting for this class of knots. These results appeal to surprisingly deep mathematics!

- **Unit 3: Knots and continued fractions**
  The relevant sections from Adams are 2.1, 2.3, 3.1, and 3.2.
- Lecture 8 (October 1)
  Tangles
  Tangles provide the building blocks for knots that Conway used for tabulation. This led us to consider some closely related (but distinct!) pairs knots related by an operation called mutation. Here is a link to a nice article in Quanta about Lisa Piccirillo's recent work on the subject.

  Lecture 9 (October 3)

Bridges
We considered a third geometric/numerical invariant called the bridge number. This has a geometric definition in terms of local maxima along a knot, but it is computable by considering an equivalent definition that counts the number of maximal overpasses in a diagram.

Lecture 10 (October 8)
Rational knots
We saw that rational knots are equivalent to two-bridge knots, and more, we saw that rational tangles were in one-to-one correspondence with rational numbers, by using continued fractions to convert a rational tangle into a fraction.

- **Assignment 3** (linked here) is due Friday October 18 by 10:59pm.

- **Unit 4: Polynomial invariants**
  The relevant section from Adams is 6.1.
- Lecture 11 (October 10)
  From diagrams to polynomials
  Today we derived a divide-and-conquer strategy for converting a diagram into a Laurent polynomial with integer coefficients.

  Lecture 12 (October 15)
  Bracket polynomials
  We found that the bracket polynomial is *almost* a knot invariant, in the sense that two diagrams for the same knot must have the same bracket polynomial up to possibly multiplying by a monomial with coefficient ±1.

  Lecture 13 (October 17)
  Trefoils
  By appealing to brackets, we showed that the left- and right-hand trefoils are distinct knots. This is a historically significant calculation!

Lecture 14 (October 22)
The Jones polynomial
There is a mild adjustment, appealing the the write of an oriented diagram, that converts the braket polynomial into a famous knot invariant known as the Jones polynomial.

- **Assignment 4** (linked here) is due Friday November 1 by 10:59pm.

- **Unit 5: Applications and further structure**
The relevant sections from Adams are 6.2 as well as parts of 6.3 and 2.3 (revisited).
- Lecture 15 (October 24)
The span of a polynomial
We introduced the span of a polynomial as the difference between the highest and lowest degrees appearing with non-zero coefficients. Using the bracket polynomial, this gives rise to to a new numerical knot invariant.

Lecture 16 (October 29)
Alternating diagrams
We proved that the span of the bracket of a reduced alternating diagram is 4 times the crossing number of the diagram. As a consequence: two reduced alternating diagrams for the same knot have the same number of crossings.

Lecture 17 (October 31)
Mutation
Mutation is the result of removing a tangle, rotating it by 180 degrees, and replacing it back in the knot diagram. This process tends to change the knot type, but this change is very difficult to detect in practice. In particular, the bracket polynomial is unable to see the difference, and today we looked closely at why this is the case. Along the way we saw that, for example, moving a small ring from one component to another (within a given link diagram) is a process that

goes undetected through the lens of the bracket polynomial. (Indeed, you can check that this s a special case of mutation.) In order to formalize all of this, we are led to consider the Kaufman bracket skein module for tangles, which we introduced today and will look at more closely next time. For people who would like to read more, a great reference is the paper *The quest for a knot with trivial Jones polynomial* by UBC's own Dale Rolfsen. You can find it through the UBC library, which has online access to the book Topics in knot theory containing the paper (it's on pages 195–210). It's an interesting read about a problem that is still open, and the discussion on skein modules is in the first few pages.

- **Assignment 5** (linked here) is due Friday November 15 by 10:59pm.

- **Unit 5: Surfaces and the Alexander polynomial**
  The relevant sections from Adams are 4.1 through 4.3 as well as parts of 6.3.
- Lecture 18 (November 5)
  The Alexander polynomial
  We introduced a second polynomial invariants of knots: the Alexander polynomial. Perhaps surprisingly, it predates the discovery of the Jones polynomial by over 50 years.

  Lecture 19 (November 7)
  Surfaces The origins of the Alexander polynomial are more geometric in origin than that of the Jones polynomial. To describe this, we needed a primer on surfaces.

  Lecture 20 (November 14)
  Seifert's algoritm (guest lecture by Pamela Shah)
  Every knot bounds an orientable surface in 3-dimensional space, and Seifert gave an algorithm for constructing such a surface that starts from a knot diagram. This gives an assignment of surfaces to diagrams, and leads to a definition of the Seifert genus of a knot: the

smallest genus among all surfaces constructed for various choices of diagram for a given knot.

Lecture 21 (November 19)
Linking and the linking matrix
Today we put our work on surfaces in conversation with the Alexander polynomial. We found a matrix, which tracks linking data from the surface, and used this to derive the Alexander polynomial by calculating a determinant. This is a little surprising, given how we came to it, but it hints at the geometric/topological origins of this polynomial going back to Alexander's original work. For those interested in learning more, I strongly encourage you to take Math 426 in the future!

Lecture 22 (November 21)
Linking and the Alexander polynomial
The end of last lecture was a little rushed. Today we went through the final calculation a little more slowly, and considered a more complicated example too.
- **Assignment 6** (linked here) is due Friday November 29 by 10:59pm.

- **Unit 7: Braids**
The relevant section from Adams is 5.4.
- Lecture 23 (November 26)
The braid group
Today we changed focus from knots and links to braids, and saw that this new class of objects forms a group.

Lecture 24 (November 28)
Markov's theorem
A theorem of Alexander states that every knot can be represented as the closure of a braid. This leads to the question of when two braids, possibly distinct, give rise to the same knot or link when closed. The answer is given by the Markov moves, which tell us how to relate braids

with equivalent closures.

Lecture 25 (December 3)

Artin Combing

The trivial knot is a hard thing to detect, but owing to the additional structure of braids the trivial braid is easily detectable. Artin showed how to do this using combing. From our point of view, this is a strategy that first considers the permutation associated with a braid, and then in the case where this permutation is trivial, considers twisting among some subset of the strands. For those of you with a background in (or those planning to take a course on) group theory, Artin's construction gives an excellent example of a semi-direct product of groups.