Final Exam

Monday, December 9, 2013

No books, notes or calculators

Problem 1.
Define the following terms:
(a) transcendental field extension
(b) automorphism group of a field extension
(c) minimal polynomial
(d) Galois extension
(e) separable polynomial
(f) separable field extension

Problem 2.
Carefully state:
(a) theorem of the primitive element
(b) a result describing the orbits of an action of the Galois group of the splitting field of a polynomial,
(c) fundamental theorem of Galois theory

Problem 3.
Give examples of the following phenomena (without proofs):
(a) a field with 9 elements,
(b) a field extension of degree 3 which is Galois,
(c) a field extension of degree 2 which is not Galois,
(d) a field extensions of degree 52,
(e) a Polynomial whose Galois group is S_5,
(f) an inseparable field extension,

Problem 4.
Prove that there is no field with exactly 15 elements.

Problem 5.
Prove that $x^4 - 18x^2 + 6$ is irreducible in $\mathbb{Q}[x]$, and find the degree of its splitting field over $\mathbb{Q}$, and its Galois group.

Problem 6.
Let L/K be an algebraic field extension, and let $R \subset L$ be a ring, which contains K. Prove that R is a field.

Problem 7.
Prove that $\cos \frac{2\pi}{5}$ is algebraic and find its minimal polynomial. Do the same for $\sin \frac{2\pi}{5}$.

Problem 8.
Show that $\cos \frac{\pi}{12} \in \mathbb{Q}(\sqrt{2} + \sqrt{3})$.