Math 422/501, Winter 2006, Term 1

Final Exam

Tuesday, December 12, 2006

No books, notes or calculators

Problem 1.
Define the following terms:

(a) centre of a group
(b) normal subgroup
(c) solvable group
(d) Sylow subgroup
(e) minimal polynomial
(f) Galois extension
(g) separable polynomial

Problem 2.
Carefully state:

(a) the orbit equation,
(b) a result describing the orbits of an action of the Galois group of the splitting field of a polynomial,
(c) a criterion by which to recognize Galois extensions.

Problem 3.
Give examples of the following phenomena (without proofs):

(a) a non-trivial normal subgroup of the group of isometries of the Euclidean plane,
(b) a Sylow 5-subgroup which is not normal,
(c) a finite group which is not solvable,
(d) a field with 9 elements,
(e) a field extension of degree 3 which is not Galois,
(f) a field extension of degree 2 which is not Galois,
(g) a Galois extension with cyclic Galois group of order 3.

Problem 4.
Let $(\mathbb{Z}/n\mathbb{Z})^*$ be the multiplicative group of the ring $\mathbb{Z}/n\mathbb{Z}$. In other words, $(\mathbb{Z}/n\mathbb{Z})^*$ is the group of residue classes $a + n\mathbb{Z}$ modulo n, such that a is coprime to n, with multiplication as group operation. Let G be a cyclic group of order n. Prove that $\mathrm{Aut}(G)$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^*$.

Problem 5.
Let G be the group of rotational symmetries of the cube.
(a) Prove that G contains 24 elements.
(b) How many Sylow 3-subgroups does G contain? Describe them geometrically.
(c) How many Sylow 2-subgroups does G contain? Describe them geometrically.

Problem 6.
Prove that the fields $\mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$ and $\mathbb{Q}(\sqrt{3}) \subset \mathbb{R}$ are not isomorphic.

Problem 7.
Let K be a field and assume that the characteristic of K does not divide the integer $n > 0$. Let L/K be a splitting field of the polynomial $x^n - 1 \in K[x]$. Denote by $\mu_n \subset L$ the group of n-th roots of unity in L.
(a) Prove that L/K is a Galois extension. Let G be the Galois group.
(b) Construct an injective homomorphism of groups $G \to \mathrm{Aut}(\mu_n)$.
(c) Conclude that G is abelian.

Problem 8.
Let $\zeta = e^{2\pi i/12}$ and $K = \mathbb{Q}(\zeta) \subset \mathbb{C}$.
(a) Prove that $K/\mathbb{Q}$ is a Galois extension. Let G be the Galois group.
(b) Prove that G is isomorphic to the Kleinian 4-group.
(c) Find the minimal polynomial of ζ over $\mathbb{Q}$.
(d) Factor the polynomial $x^{12} - 1 \in \mathbb{Q}[x]$ into irreducible factors.