The University of British Columbia
Final Examination - April 12 , 2006

Mathematics 421/510, Real Analysis II, Term 2

Instructor: Dr. Brydges

Closed book examination Time: 2.5 hours

Special Instructions:
- This exam has five questions

1. Let Y be a topological space and let A be a set. Let $Y^A = \{f : A \to Y\}$ be the product space $\prod_{\alpha \in A} Y$ with the product topology.

   (a) The product topology on Y^A is the weakest topology such that ...?

   (b) Describe a neighbourhood base for a point $f \in Y^A$.

   (c) Show that pointwise convergence, $f_n(\alpha) \to f(\alpha)$ for each $\alpha \in A$, implies $f_n \to f$.

   (d) Does every sequence $\{f_n\}$ with $f_n \in \{0,1\}^{[0,1)}$ have a convergent subsequence? (Yes/No plus very brief comment in either case).

2. Let $\mathcal{X}$ be a normed vector space over the complex numbers and let $\mathcal{X}^*$ be the space of continuous linear functionals on $\mathcal{X}$.

   (a) Define the norm $\|f\|$ of $f \in \mathcal{X}^*$.

   (b) State the complex version of the Hahn Banach theorem.

   (c) Let $x_0 \in \mathcal{X}$. Show that there is a linear functional $f \in \mathcal{X}^*$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$.

   (d) Suppose that $x_n \to x$ weakly. Prove that $\|x\| \leq \liminf \|x_n\|$.

   (e) Suppose that $\mathcal{X}$ is a Hilbert space, that $x_n \to x$ weakly and $\|x\| = \lim \|x_n\|$. Prove that $x_n \to x$ in norm.

   (f) Is it possible for $x_n \to x$ weakly and $\|x\| < \liminf \|x_n\|$? Hint: Bessel inequality.

3. (a) Are continuous functions dense in $L^\infty([0,1], dx)$? (Yes/No plus brief explanation in either case).

(b) Define the term *complete* orthonormal set (orthonormal basis) in the context of a separable Hilbert space.

(c) Prove that if $f \perp \mathcal{D}$ where $\mathcal{D}$ is a dense subset of a Hilbert space, then $f = 0$.

(d) For $k \in \mathbb{Z}$ and $x \in [0, 2\pi]$, let $e_k(x) = (2\pi)^{-1/2} e^{ikx}$. You may assume these functions are an orthonormal set in $L^2([0, 2\pi])$ and that continuous functions compactly supported in $(0, 2\pi)$ are dense in $L^2([0, 2\pi])$. Prove that $\{e_k\}$ is a *complete* orthonormal set in $L^2([0, 2\pi])$.

4. Let $\mathcal{X}$ be a Banach space, let $\{T_n\} \in L(\mathcal{X}, \mathcal{X})$ be a sequence of continuous linear operators on $\mathcal{X}$.

(a) There are at least three notions of convergence for the sequence T_n. What are they?

(b) Suppose, $\forall x \in \mathcal{X}, \forall f \in \mathcal{X}^*$, that $f(T_n x) \to f(Tx)$ where T is a linear operator. Show that $T \in L(\mathcal{X}, \mathcal{X})$.

5. Let $T \in L(\mathcal{X}, \mathcal{X})$, where $\mathcal{X}$ is a Banach space.

(a) Define the resolvent set $\rho(T)$ and the resolvent R_λ of T.

(b) Prove that

$$T = \frac{1}{2\pi i} \oint_\Gamma R_\lambda \lambda \, d\lambda,$$

where Γ is the oriented boundary of an open disk $D \supset \sigma(T)$.