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The University of British Columbia

Sessional Exams – 2007-08 Term 2
Mathematics 419 Stochastic Processes
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This exam consists of **7** questions worth **10** marks each.
No calculators or other aids are permitted.
Show all work and calculations and explain your reasoning thoroughly.

1. Consider simple random walk on $\mathbb{Z}^3$, taking steps $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ with probabilities $\frac{1}{6}$.

   (a) Let $p_{00}(2n)$ denote the probability that the walk starting at the origin returns to the origin in $2n$ steps. Show that

   $$p_{00}(2n) = \frac{1}{2^{2n}} \binom{2n}{n} \sum_{j,k \geq 0 : j+k \leq n} \left[\frac{1}{3^n} \binom{n}{j,\ k,\ n-j-k} \right]^2,$$

   where the multinomial coefficient $\binom{n}{j,k,l}$ is defined, for non-negative integers j, k, l with $j + k + l = n$, by $\binom{n}{j,k,l} = \frac{n!}{j!k!l!}$. Explain in complete detail how you obtained this result.

   (b) Show that for any $j, k, l \geq 0$ such that $j + k + l = n$,

   $$\binom{n}{j,\ k,\ l} \leq \frac{n!}{\lfloor n/3 \rfloor^3},$$

   where $\lfloor n/3 \rfloor$ denotes the greatest integer less than or equal to $n/3$.

   (c) Show that the walk is transient. (Recall Stirling's formula $n! \sim n^n e^{-n} \sqrt{2\pi n}$, and use the inequality of (b) on just one of the factors on the right-hand side of (a).)

2. At all times, an urn contains N balls, some black and some white. At each stage, a coin having probability $p \in (0, 1)$ of landing heads is flipped. If heads appears, then a ball is chosen at random from the urn and is replaced by a white ball. If tails appears, then a ball is chosen at random from the urn and is replaced by a black ball. Let X_n denote the number of white balls in the urn after the nth stage.

   (a) What are the transition probabilities of this chain?

   (b) Explain why the Markov chain has a unique stationary distribution.

   (c) Find the stationary distribution.

   (d) Show that the chain is reversible.

3. A viral linear DNA molecule of length 1 is often known to contain a certain "marked position," with the exact location of this mark being unknown. One approach to locating the marked position is to cut the molecule by agents that break it at points chosen according to a Poisson process with rate λ. It is then possible to determine the fragment that contains the marked position. For instance, letting m denote the location on the line of the marked position, then if L_1 denotes the last Poisson event time before m (or 0 if there are no Poisson events in $[0, m]$), and R_1 denotes the first Poisson event time after m (or 1 if there are no Poisson events in $[m, 1]$), then it would be learned that the marked position lies between L_1 and R_1.

   (a) Find $P(L_1 = 0)$.

   (b) Find $P(L_1 < x)$ for $0 < x < m$.

   (c) Find $P(R_1 = 1)$.

   (d) Find $P(R_1 > x)$ for $(m < x < 1)$.

   (e) By repeating the preceding process on identical copies of the DNA molecule, we are able to zero in on the location of the marked position. If the cutting process is utilized on n

identical copies of the molecule, yielding the data L_i, R_i, $i = 1, \dots, n$, then it follows that the marked position lies between L and R, where

$$L = \max_i L_i, \quad R = \min_i R_i.$$

Find ER and EL and show that, for large n, $E[R - L] \sim 2/(n\lambda)$.

4. This problem shows that the bounded convergence theorem remains true when almost sure convergence is weakened to convergence in probability.

   Let $X_1, X_2, \dots$ be a sequence of random variables, and let X be a random variable. Suppose that $X_n \to X$ in probability. Suppose that there is a finite positive constant M such that $|X_n(\omega)| \le M$ uniformly in all n and ω. Conclude that $E|X_n - X| \to 0$, i.e., $X_n \to X$ in L^1.

   Hint: you may find it useful to argue first that $P(|X| > M + m^{-1}) = 0$ for all positive integers m.

5. Consider the following simple model for emptying a queue. Let $\xi_1, \xi_2, \dots$ be i.i.d. Poisson random variables with mean μ. There are X_n people in the queue at time n. If $X_n \ge 1$, then one person is served and another ξ_{n+1} people enter the queue. Thus $X_{n+1} = X_n - 1 + \xi_{n+1}$ when $X_n \ge 1$. If $X_n = 0$ then the queue is empty, work stops, and $X_{n+1} = 0$. Initially $X_0 = x \ge 1$.

   (a) When is the queue X_n a supermartingale, a submartingale, a martingale? Explain why X_n must converge to 0 a.s. when $\mu \le 1$, and must converge either to 0 or $+\infty$ when $\mu > 1$.

   (b) Let $\mu > 1$. Use calculus to check that the equation $E\rho^{\xi_n} = \rho$ has a solution $\rho < 1$.

   (c) Let $\mu > 1$. Show that ρ^{X_n} is a martingale.

   (d) Let $\mu > 1$. Using the above martingale, determine the probability that the queue ever empties (i.e., that X_n is eventually zero).

6. Recall that for Brownian motion, if T is a bounded stopping time then $EW(T) = EW(0)$. Let T_x be the hitting time of $x \in \mathbb{R}$ for Brownian motion started from 0. For $a < 0 < b$, prove that $P(T_a < T_b) = b/(b - a)$.

7. Let W be a standard Brownian motion with $W(0) = 0$. Let $0 < s < t$.

   (a) What is the distribution of $W(s) + W(t)$?

   (b) Show that the conditional distribution of $W(s)$ given that $W(t) = b$ is $N(\frac{s}{t}b, \frac{s}{t}(t - s))$.