Be sure this exam has 3 pages including the cover

The University of British Columbia

Sessional Exams – 2006-07 Term 2
Mathematics 419 Stochastic Processes
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This exam consists of **7** questions worth **10** marks each.
No calculators or other aids are permitted.
Show all work and calculations and explain your reasoning thoroughly.

1. Each candidate should be prepared to produce his library/AMS card upon request.

2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

1. Consider the symmetric simple random walk on $\mathbb{Z}^2$, started from the origin. This walk takes independent steps $(\pm 1, 0)$, $(0, \pm 1)$ with equal probabilities $\frac{1}{4}$.
   (a) Let p_{2n} be the probability that the walk is at the origin at time $2n$. Prove that
   $$p_{2n} = \frac{1}{4^{2n}} \binom{2n}{n}^2.$$
   (b) Prove that the walk is recurrent.

2. A collection of $2N$ balls, N of which are black and N white, are placed in two urns in such a way that each urn contains N balls. At each time step, one ball is selected at random from each urn and the two selected balls are interchanged. The state of the system is given by the number of black balls in the first urn.
   (a) What are the transition probabilities for this Markov chain?
   (b) Explain why this Markov chain will converge to an equilibrium.
   (c) Determine the stationary distribution and prove that it is the unique stationary distribution.

3. Let $(N(t))_{t\geq 0}$ be a Poisson process with rate λ, and let T be an exponential random variable with mean $1/\mu$, with T independent of the Poisson process. Determine the distribution of the random variable $N(T)$. (You may need to know that $\int_0^\infty x^n e^{-x} dx = n!$.)

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $X_n \to X$ in probability. Prove that $f(X_n) \to f(X)$ in probability. Hint: Consider separately the cases $P(|X| > A)$ and $P(|X| \leq A)$, with A large, and restrict attention to a suitable interval on which f is uniformly continuous.

5. Let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$, where the X_n are i.i.d. and equal to $+1$ or -1 with probability $\frac{1}{2}$ each. Let $r, s > 0$. Let T be the hitting time of $-s$ or r, i.e., T is the smallest value of n such that $S_n \in \{-s, r\}$.
   (a) Show that S_n is a martingale. Using this martingale stopped at T, compute ES_T and hence $P(S_T = r)$.
   (b) Show that $S_n^2 - n$ is a martingale. Using this martingale stopped at T, compute ET.

6. (a) Let $(X_n)_{n\geq 1}$ be a sequence of random variables and assume that $\sup_n E[X_n^2] < C$ for some finite constant C. Prove that $(X_n)_{n\geq 1}$ is uniformly integrable.
   (b) A certain insurance company has an equal chance of collecting a premium or paying a claim each day on which there is activity. However some days there is no activity at all. It starts with an initial reserve of $X_0 = x \geq 1$, where x is a (non-random) integer. If it has a reserve $X_n \geq 1$ on day n, then at the next day it will gain or lose an amount Y_{n+1}, where the Y_i are independent random variables with $Y_{n+1} = 1, 0, -1$ with respective probabilities $\frac{1}{2}p_n, (1 - p_n), \frac{1}{2}p_n$. Thus $X_{n+1} = X_n + Y_{n+1}$ as long as $X_n \geq 1$. If $X_n = 0$ then the company is declared bankrupt and $X_{n+1} = 0$.
   Prove that $EX_n = x$ for all $n \geq 1$ and that X_n converges to a random variable X_∞ a.s.
   (c) If $\sum_n p_n < \infty$ (i.e., there is rapidly decreasing activity), show that $EX_\infty = x$. Hint: part (a) may be helpful here.

(d) Suppose instead that $\sum_n p_n = \infty$. What is EX_∞? (It is not x.)

7. In this problem, express your answers in terms of the c.d.f. $\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-1/2} e^{-t^2/2} dt$ of a standard normal random variable.
Mary and June are in a bicycle race. Let $Y(t)$ denote the time in seconds by which Mary is ahead when $100t\%$ of the race has been completed. Suppose that $(Y(t))_{0\le t\le 1}$ is a standard Brownian motion.

   (a) Suppose that Mary leads by 1 second at the midpoint of the race. What is the probability that she is the winner?

   (b) Recall from Assignment 9 that if $0 < s < t < 1$ then the conditional distribution of $Y(s)$ given that $Y(t) = b$ is $N(\frac{s}{t}b, \frac{s}{t}(t-s))$. Suppose that Mary wins the race by a margin of 1 second. What is the probability that she was ahead at the midpoint of the race?