Math 405/607E (Wetton) Final Exam, Fall 2010

Instructions:

- No calculators or notes permitted.
- Show all work in booklets.
- Do eight questions:
  - All six questions in part A (#A1-A6)
  - Two of four questions in part B (#B1-B4).
- All questions worth 10 marks (80 points total).

Part A (#1-6): Do all six questions

A1. The following statements are all *false*. Briefly describe a counter-example (one sentence, no technical details are necessary).

(a) To multiply a vector by $\mathbf{A}^{-1}$ you must always first find all the entries of $\mathbf{A}^{-1}$.

(b) Multiplication by $\mathbf{A}$ can only be done quickly if $\mathbf{A}$ is sparse.

(c) All consistent multi-step methods converge.

(d) Any implicit time-stepping method can be used on any problem with no time-step restriction due to stability.

(e) Conjugate gradient methods are always faster than a direct method (sparse gaussian elimination) at solving systems of equations when the coefficient matrix $\mathbf{A}$ is sparse.

A2. It is known that $f(0) = 1$, $f'(0) = 1$ and $f'(1) = 3/2$. Use quadratic interpolation of this data to estimate $f(1/2)$. *Note:* one of your options in part B is to find the error in this approximate value.

A3. Consider the Leap-Frog time stepping scheme for $\dot{y} = f(y, t)$:

$$Y^{n+1} = Y^{n-1} + 2kf(Y^n, nk)$$

(a) [5 marks] Show that the scheme is second order accurate.

(b) [5] Write the quadratic equation that describes the growth factors $G(z)$ where $z = k\lambda$ and the method is applied to the problem with $f(y,t) = \lambda y$.

Note: one of your options in part B is to find the stability region of this method.

A4. Consider the Lax Wendroff scheme applied to the problem $u_t = u_x$:

$$U^{n+1} = U^n + kD_1U^n + \frac{k^2}{2}D_2U^n$$

where solutions are 1-periodic in x. Consider the scheme with $k = Ch$ with $0 < C < 1$. Show that it is second order accurate. *Hint:* You will have to differentiate the partial differential equation with time to match some of the terms.

A5. For the Lax Wendroff scheme above in #A4, show that the method is stable as long as $k < h$. *Hint:* use von-Neumann analysis.

A6. Consider the discretized fourth order boundary value problem

$$D_4U_j + U_j = F_j$$

in the 1-period setting with

$$D_4U_j = \frac{1}{h^4}(U_{j-2} - 4U_{j-1} + 6U_j - 4U_{j+1} + U_{j+2})$$

Show that undamped Jacobi iterations applied to this problem do not converge. *Hint:* use von-Neumann analysis and recall that $\cos(2\theta) = 2\cos^2\theta - 1$.

Part B (#B1-B3): Do two of the three questions

B1. Continue problem #A2. It is known that $|f'''(x)| < 0.1$ for all $x \in [0,1]$. Give a sharp bound on the error in your answer to #A2.

B2. Find the stability region for the leap-frog time stepping scheme in problem #A3.

B3. Consider the following nonlinear boundary value problem for $u(x)$:

$$-u'' + u + (u_x)^2 = f(x)$$

with boundary conditions $u(0) = 0$ and $u(1) = 0$.

(a) [4 points] Write down a second order finite difference approximation of this problem.

(b) [6] Write down the entries of the Jacobian derivative matrix of your discretization.

B4. Consider the method of lines problem for vectors $U(t)$ and $V(t)$ on the a spatially 1-periodic grid with

$$\dot{U} = V \quad \text{and} \quad \dot{V} = D_2 U$$

(a discretization of the wave equation) with given initial conditions U_0 and V_0. Show the stability of the method, *i.e.* show that

$$\|U(t)\| \leq C(\|U_0\| + \|V_0\|) \quad \text{and} \quad \|V(t)\| \leq C(\|U_0\| + \|V_0\|)$$

with a constant C that may depend on t but not h (the exact norms above are not specified). This is a difficult problem, but any progress will be given generous part marks.