This examination has 5 questions and 3 pages.

The University of British Columbia
Final Examinations—December 2006

Mathematics 403
Stabilization and Optimal Control of Dynamical Systems (*Professor Loewen*)

Open book examination. Time: $2\frac{1}{2}$ hours

Any resources used in class may be used during the examination.
Write your answers in the official examination booklet. Start each solution on a separate page.

[15] **1.** Consider the matrix below, where $\omega > 0$ is an unknown constant and $\zeta = \sqrt{3}/2$:

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\omega\zeta \end{bmatrix}.$$

(a) Find e^{tA}.

(b) Find $e^{t\widetilde{A}}$, where $\widetilde{A} = P^{-1}AP$ and $P = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$.

(c) Show that for each piecewise continuous function $u\colon [0, +\infty) \to \mathbb{R}$ that satisfies $|u(t)| \leq 10^{403}$ for almost all t, and each $x\colon [0, +\infty) \to \mathbb{R}$ obeying

$$\ddot{x}(t) + 2\omega\zeta\dot{x}(t) + \omega^2 x(t) = u(t) \qquad \text{a.e. } t \in [0, +\infty),$$

one has $\sup_{t \geq 0} |x(t)| < \infty$.

[20] **2.** Consider the single-input system $\dot{x} = Ax + Bu$ in which

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}. \qquad (*)$$

It can be shown that $\det(sI - A) = s^4 + s^3 - 2s^2 - s$.

(a) Show that the system $(*)$ is controllable.

(b) Find an invertible matrix P such that the pair $(\widetilde{A}, \widetilde{B}) = (P^{-1}AP, P^{-1}B)$ has controllable canonical form.

(c) Find a feedback matrix F for which the four eigenvalues of $A + BF$ are $-1, -1 \pm i, -2$.

[20] **3.** Let $\mathcal{A}(T)$ denote the set of all vectors $x(T)$ corresponding to solutions for this system:

$$\begin{aligned} \dot{x}_1(t) &= \quad x_2(t), & x_1(0) &= 0, \\ \dot{x}_2(t) &= -x_1(t) \qquad + u(t), & x_2(0) &= 0, \\ |u(t)| &\leq 1. \end{aligned}$$

(a) Find and sketch $\mathcal{A}(\pi/2)$.

(b) Find and sketch $\mathcal{A}(\pi)$.

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[20] **4.** In this problem, the state x and control u are scalars, α, β, A, and B are constants, and $A < B$:

$$\begin{aligned}
\text{minimize} \quad & \int_0^1 \left(\tfrac{1}{2}u(t)^2 + e^{-x(t)}\right) dt \\
\text{subject to} \quad & \dot{x}(t) = e^{x(t)}u(t) && \text{a.e. } t \in [0,1], \\
& u(t) \in [A, B] && \text{a.e. } t \in [0,1], \\
& x(0) = \alpha, \ x(1) = \beta.
\end{aligned}$$

(a) Show that the dynamics $\dot{x} = ue^x$ and endpoint conditions $x(0) = \alpha$, $x(1) = \beta$ can all be satisfied with a *constant* control, $u(t) \equiv c$. Express c in terms of α and β.

(b) Assume that the control constraints obey $A < c < B$, taking c from part (a). Show that every extremal control $u(\cdot)$ is both *nonconstant* and *nonincreasing*. Give a qualitative description of the form of a typical extremal control.

(c) Discard the constraint "$u \in [A, B]$" and find the unique extremal control-state pair in terms of β, assuming $\alpha = 0$.

[25] **5.** Consider the following optimal control problem with scalar state x and control u:

$$\begin{aligned}
\text{minimize} \quad & 3x(\pi)^2 + \int_\tau^\pi u(t)^2 \, dt \\
\text{subject to} \quad & \dot{x}(t) = (\pi - t)u(t), && \text{a.e. } t \in (\tau, \pi), \\
& u(t) \in \mathbb{R}, && \text{a.e. } t \in (\tau, \pi), \\
& x(\tau) = \xi.
\end{aligned}$$

Solve the following parts in whatever order you find most convenient.

(a) Find an extremal control-state pair in terms of the initial point (τ, ξ), assuming $\tau < \pi$.

(b) Show that the extremal in (a) is a true minimizer.

(c) Find the true Hamiltonian, $\mathbb{H}(t, x, p)$, for this problem.

(d) Find a function $v = v(t, x)$ that satisfies

$$\begin{aligned}
& v_t(t, x) + \mathbb{H}(t, x, -v_x(t, x)) = 0, && 0 < t < \pi, \ x \in \mathbb{R}, \\
& v(\pi, x) = 3x^2, && x \in \mathbb{R}.
\end{aligned}$$

(e) Find an optimal control law in feedback form. That is, find a function $U = U(t, x)$ such that for each (τ, ξ) with $\tau < \pi$, the unique solution $x(\cdot)$ of

$$\dot{x}(t) = (\pi - t)U(t, x(t)), \quad \text{a.e. } \tau < t < \pi, \qquad x(\tau) = \xi,$$

is the extremal arc identified in part (a).

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Selected Formulas

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$|pq| \le \frac{1}{2}\left(p^2+q^2\right)$$

$$\dot{x} = Ax + Bu \implies x(t) = e^{A(t-r)}x(r) + \int_r^t e^{A(t-s)}Bu(s)\,ds$$

$(I-M)^{-1} = I + M + M^2 + \cdots$ for any square matrix M such that the right side converges

$$\omega \neq 0,\ \gamma^2 \neq \omega^2,\ X(t) = \frac{\beta\sin(\gamma t)}{\omega^2-\gamma^2} + \frac{\alpha\cos(\gamma t)}{\omega^2-\gamma^2} \implies \ddot{X}(t) + \omega^2 X(t) = \alpha\cos(\gamma t) + \beta\sin(\gamma t)$$

$$\omega \neq 0, \qquad X(t) = \frac{\alpha t\sin(\omega t)}{2\omega} - \frac{\beta t\cos(\omega t)}{2\omega} \implies \ddot{X}(t) + \omega^2 X(t) = \alpha\cos(\omega t) + \beta\sin(\omega t)$$