This examination has 5 questions on 5 pages.

The University of British Columbia
Final Examinations—December 2007

Mathematics 402
Calculus of Variations (*Professor Loewen*)

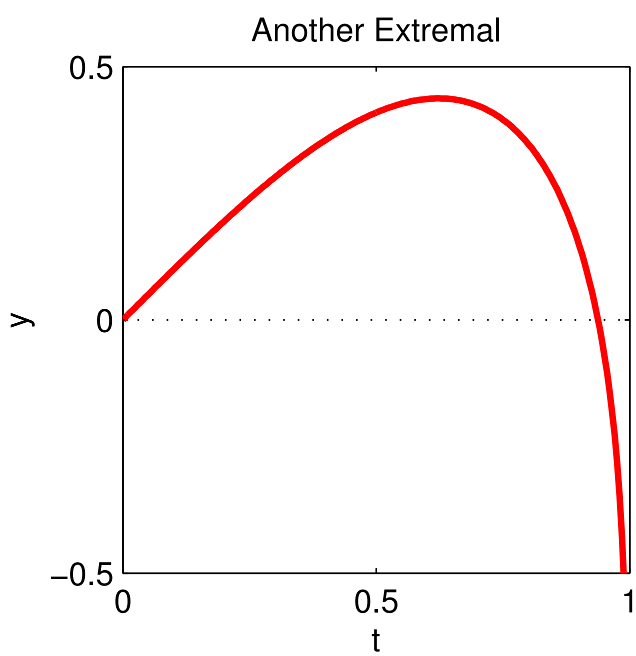
Duration:	150 minutes.
Permitted:	Any handwritten or instructor-generated materials.
Forbidden:	All electronic devices.
Presentation:	Write your answers on the coloured sheets provided. Start each question on a new sheet.

[20] **1.** Let $L(t,x,v) = (1-t^2)v^2 + \lambda x^2$.

(a) Find all constants λ, k, c, that make $x(t) = t^2 + kt + c$ an extremal for L.

(b) Find another value of λ for which L has a nonconstant polynomial extremal. (Find the extremal, too.)

(c) Take λ, k, c, and $x(\cdot)$ from part (a). For each fixed $b > 0$, decide whether the extremal x itself provides the optimal path from $(0, x(0))$ to $(b, x(b))$. Address all possible types of local minimality: directional, weak, and strong. (Expect different answers for different values of b.)
Hint: the arc y sketched below is an extremal for L obeying $y(0) = 0$, $\dot{y}(0) = 1$.



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[20] **2.** (a) Solve the following problem. Note that $x(2)$ may take any real value.

$$\min_{x \in PWS[1,2]} \left\{ 4x(2)^2 + \int_1^2 \left(t^2 \dot{x}(t)^2 + 2x(t)^2 \right) dt \,:\, x(1) = 1 \right\}.$$

(b) Among all arcs x in $PWS[1,2]$ obeying both

$$x(1) = 1 \qquad \text{and} \qquad 24x(2)^2 + \int_1^2 12x(t)^2\, dt = 5,$$

identify the one that minimizes $M[x] \stackrel{\text{def}}{=} \int_1^2 t^2 \dot{x}(t)^2\, dt.$

Hint: The result from part (a) should help with part (b).

[20] **3.** Consider the following minimization problem, in which α is a constant:

$$\min_{x \in PWS[0,1]} \left\{ \Lambda[x] := \int_0^1 \left(\sqrt{1 + \dot{x}(t)^2} + \alpha x(t) \right) \, dt \; : \; x(0) = 1, \; x(1) = 1 \right\}.$$

(a) Explain why any local minimizer must be C^2.

(b) Prove: If $\alpha > 0$ and $x(\cdot)$ is an extremal, then $x(\cdot)$ must be a convex function. What happens when $\alpha < 0$? When $\alpha = 0$?

(c) When $\alpha = 2007$, the problem has no minimum. Prove this by constructing a sequence of admissible arcs $x_1, x_2, \ldots$, such that $\Lambda[x_n] \to -\infty$ as $n \to \infty$.

(d) Find the unique global solution in two cases:

(i) $\alpha = 2/\sqrt{5}$, (ii) $\alpha = -2/\sqrt{5}$.

Suggestion: Work algebraically with general α as much as possible. That way every step can help you twice.

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[20] **4.** Let $L(t, x, v) = v^2 + 4xv + \alpha x^2 + 2tx$, $(a, A) = (0, 0)$, and $(b, B) = (1, 1)$ in the basic problem.

(a) Find all real values of the parameter α for which an admissible extremal exists.

(b) For every such extremal, classify it as either Extremal, Weak Local Minimizer, Strong Local Minimizer, or Global Minimizer. (Make the strongest statement you can justify.)

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[20] **5.** In a "point-to-curve" problem for $L = L(t, x, v)$, the initial point (a, A) is given but the final point is free to vary along some curve C in (t, x)-space. When C is given parametrically,[1] i.e.,

$$C = \{(b, B) = (b(\theta), B(\theta)) \,:\, 0 < \theta < 1\},$$

the problem becomes

$$\min_{\theta, x(\cdot)} \left\{ \int_a^{b(\theta)} L(t, x(t), \dot{x}(t))\, dt \,:\, x(a) = A,\ x \in PWS[a, b(\theta)],\ x(b(\theta)) = B(\theta) \right\}. \quad \text{(P)}$$

Recall the value function defined below; assume it is continuously differentiable:

$$V(T, X) = \min_{x(\cdot)} \left\{ \int_a^{T} L(t, x(t), \dot{x}(t))\, dt \,:\, x(a) = A,\ x \in PWS[a, T],\ x(T) = X \right\}.$$

(a) Suppose the parameter $\theta = \widehat{\theta}$ and arc $\widehat{x}$ give the minimum in problem (P). Write $\widehat{b} = b(\widehat{\theta})$, $\widehat{B} = B(\widehat{\theta})$, $\widehat{L}(t) = L(t, \widehat{x}(t), \dot{\widehat{x}}(t))$, etc. Explain why these two vectors in (t, x)-space must be perpendicular:

$$\left(V_t(\widehat{b}, \widehat{B}), V_x(\widehat{b}, \widehat{B})\right), \quad \left(b'(\widehat{\theta}),\, B'(\widehat{\theta})\right).$$

(b) In the situation of part (a), explain why these two vectors in (t, x)-space must be perpendicular:

$$\left(\widehat{L}(\widehat{b}) - \widehat{L}_v(\widehat{b})\dot{\widehat{x}}(\widehat{b}),\, \widehat{L}_v(\widehat{b})\right), \quad \left(b'(\widehat{\theta}),\, B'(\widehat{\theta})\right).$$

(c) Suppose $L(t, x, v) = f(t, x)\sqrt{1 + v^2}$ for some positive-valued $f = f(t, x)$. Explain why the minimizing arc $\widehat{x}$ must cross the target curve C at right angles.

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[1] Assume sufficient differentiability and $(b'(\theta), B'(\theta)) \neq (0, 0)$ for all θ.