THE UNIVERSITY OF BRITISH COLUMBIA
Sessional Examinations—April 2014

MATHEMATICS 401

TIME: 3 hours

INSTRUCTIONS: Please read carefully.

1. Try as many questions as you can. Each question is worth 25 points. It is possible to obtain a total of 125 points. A mark of N/125 will be treated as a mark of N/100.
2. Closed book exam. In particular, no notes or electronic devices are permitted.

#1. Find the eigenvalues to within 1% accuracy for the Dirichlet problem

$$\begin{aligned} &-y'' + \varepsilon x^2 y = \lambda y, \quad 0 < x < 1, \\ &y(0) = y(1) = 0, \end{aligned}$$

where $\varepsilon = \text{const}, |\varepsilon| \leq 0.15$.

#2. Consider Problem (I) given by

$$\begin{aligned} &u_t - u_{xx} = f(x,t), \quad t > 0, 0 < x < \infty, \\ &u_x(0,t) = g(t), \quad t > 0, \\ &u(x,0) = h(x), \quad 0 < x < \infty. \end{aligned}$$

(a) Set up the boundary value problem satisfied by the Green's function for Problem (I).

(b) Express the solution $u(x,t)$ of Problem (I) in terms of the given data $\{f(x,t), g(t), h(x)\}$ and the Green's function defined by the solution of the boundary value problem set up in part (a).

(c) Recall that the fundamental solution of the heat equation is given by $K(x,t) = \dfrac{e^{-x^2/4t}}{\sqrt{4\pi t}}$. Find the Green's function for Problem (I).

#3. Consider Problem (II) given by

$$q(\mathbf{x})u + \rho(\mathbf{x})\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (p(\mathbf{x})\nabla u) = f(\mathbf{x},t), \quad \mathbf{x} \in D, t > 0,$$

$$u = g(\mathbf{x},t), \quad \mathbf{x} \in \partial D, t > 0,$$

$$u(\mathbf{x},0) = \varphi(\mathbf{x}), \quad \mathbf{x} \in D,$$

$$\frac{\partial u}{\partial t}(\mathbf{x},0) = \psi(\mathbf{x}), \quad \mathbf{x} \in D,$$

where in domain *D,* one has $\rho(\mathbf{x}) > 0,\ p(\mathbf{x}) > 0,\ q(\mathbf{x}) \geq 0.$

(a) Set up the boundary value problem satisfied by the Green's function for Problem (II).
(b) Express the solution $u(x,t)$ of Problem (II) in terms of the given data $\{f(\mathbf{x},t),\ g(\mathbf{x},t),\ \varphi(\mathbf{x}),\ \psi(\mathbf{x})\}$ and the Green's function defined by the solution of the boundary value problem set up in part (a).
(c) Formally construct the Green's function of Problem (II) in terms of an appropriate series expansion. In particular, set up the problem satisfied by the eigenfunctions in the series expansion.
(d) Determine the time dependence of the coefficients of the eigenfunctions in the series expansion.
(e) Set up the variational problem to find the successive sequence of eigenvalues and eigenfunctions in the series expansion.

#4. Consider simple closed curves of fixed length ℓ. Let A be the area enclosed by any such curve. Use the calculus of variations to find the shape of the curve yielding the maximum area *A*. You do not have to prove that your extremizing curve yields a maximum area. What is the value of the maximizing area A? [Hint: Consider parametrizing the curve.]

#5. Write a short and coherent essay, explaining in words on when one can use a variational procedure to solve a posed boundary value problem for a given PDE system. In your essay, carefully state which types of PDE systems are amenable to a variational procedure. How is a variational procedure used to solve such a posed boundary value problem? How is the variational procedure used to find approximate solutions?