MATH 401 FINAL EXAM – April 13, 2012

No notes or calculators allowed. **Time: 2.5 hours.** **Total: 50 pts.**

1. Consider the ODE problem for $u(x)$:

$$\left\{ \begin{array}{c} u'' + u' - 2u = f(x), \quad 0 < x < 1 \\ u'(0) + u(0) = 0, \quad u(1) = 2 \end{array} \right. . \tag{1}$$

   (a) (4 pts.) Write down the problem satisfied by the Green's function $G_x(y) = G(x; y)$ for problem (1) (but do **not** try to solve it).

   (b) (2 pts.) Express the solution $u(x)$ of (1) in terms of $G_x(y)$.

   (c) (4 pts.) Now suppose the BC at $x = 1$ is changed to $u(1) + au'(1) = 2$. For what value of a is a solvability condition on $f(x)$ required for (1).

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2. Fix a number $q > 0$, and consider the operator $-\Delta + q$ in the plane $\mathbb{R}^2$. *For this problem it may be helpful to recall the formula for the Laplacian in polar coordinates* $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$.

   (a) (2 pts.) Write down the equation that the free-space Green's function $G_x^{\mathbb{R}^2}(y)$ for $-\Delta + q$ in $\mathbb{R}^2$ should solve.

   (b) (3 pts.) This Green's function has the form $G_x^{\mathbb{R}^2}(y) = AK_0(\lambda|y-x|)$ where $K_0(y)$ (the *modified Bessel function of order zero*) is the solution of the ODE $y^2K_0''(y) + yK_0'(y) - y^2K_0(y) = 0$ for $y > 0$ with $K_0(y) \sim -\ln y$ as $y \to 0+$. Find the numbers A and λ.

   (c) (3 pts.) Use the method of images to find the Green's function for the following Neumann boundary-value problem in the upper half-plane,

$$\begin{cases} \Delta u = qu, & x_2 > 0 \\ \frac{\partial u}{\partial x_2}(x_1, 0) = g(x_1) \end{cases}, \tag{2}$$

   and find the solution $u(x_1, x_2)$.

   (d) (2 pts.) Prove that problem (2) cannot have more than one solution $u(x)$ which decays quickly at infinity (eg. $|u(x)| \leq C/|x|$, $|\nabla u(x)| \leq C/|x|^2$).

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3. (a) (3 pts.) Write down the problem satisfied by the Green's function $G(y, \tau; x, t)$ for the following initial - boundary value problem for the heat equation on an interval:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & 0 < x < \pi,\ t > 0 \\ u(x,0) = u_0(x) & 0 \leq x \leq \pi, \\ u(0,t) = 0 & u(\pi, t) = 0 \end{cases},$$

and express the solution $u(x,t)$ in terms of the Green's function.

(b) (4 pts.) Find the Green's function as an eigenfunction expansion.

(c) (3 pts.) Suppose $u_0(x) \geq 0$, $u_0 \not\equiv 0$. Use the maximum principle for the heat equation to show that $u(x,t) > 0$ for $t > 0$.

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4. Let $D = (0,1) \times (0,1) = \{(x_1, x_2) \mid 0 < x_1 < 1,\ 0 < x_2 < 1\}$ be the unit square in $\mathbb{R}^2$, let $f(x)$ be a smooth function on D, and consider the variational problem

$$\min_{u \in C^2(D)} \int_D \left\{ \frac{1}{2}(1 + x_1)|\nabla u(x)|^2 - f(x)u(x) \right\} dx.$$

(a) (3 pts.) Determine the problem (Euler-Lagrange equation plus BCs) that a minimizing function would solve.

(b) (3 pts.) Is there a solvability condition on $f(x)$ required for the Euler-Lagrange problem to have a solution? If so, what is the effect of this condition in the original variational problem?

(c) (4 pts.) For $f(x) = x_1 x_2 - 1/4$, find an approximate minimizer, using a Rayleigh-Ritz approach, with three trial functions $v_1(x) = 1$, $v_2(x) = x_1$, $v_3(x) = x_2$.

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5. Fix $L > 0$, and let T_L denote the triangular region in the plane bounded by the x_1-axis, the x_2-axis, and the line $x_1 + x_2/L = 1$:

$$T_L = \{ \ (x_1,\ x_2) \in \mathbb{R}^2 \mid x_1 > 0,\ x_2 > 0,\ x_1 + \frac{x_2}{L} \leq 1 \ \}.$$

Let $\lambda_1(T_L)$ denote the first Dirichlet (zero BCs) eigenvalue of $-\Delta$ on T_L.

(a) (6 pts.) Find the best upper- and lower-bounds you can for $\lambda_1(T_L)$ by comparing T_L with appropriate rectangles.

(b) (1 pts.) Find $\lim_{L\to\infty} \lambda_1(T_L)$.

(c) (3 pts.) Write the variational principle for λ_1. Find an appropriate trial function, and explain how you would use it to find an upper bound for $\lambda_1(T_L)$ (but do **not** carry out the computation).

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