The University of British Columbia

Math 401 Final Examination - April 2009

Closed book exam. No notes or calculators allowed.
Answer all questions; time 2.5 hours.

1. [**17**] Consider the problem

$$\begin{aligned} L[\,\mathrm{u}] &= u'' + \lambda^2 u = f(x),\ 0 < x < 1 \\ u'(0) &= 0;\ u'(1) = 0,\ \text{with } \lambda > 0. \end{aligned}$$

(a) Find the Green's Function for the problem, $G(\xi, x)$, and hence write the general solution in terms of $G(\xi, x)$ and $f(x)$.
(b) For what values of λ is there a solvability condition on $f(x)$, and state the condition.

2. [**16**] Consider the problem

$$\begin{aligned} L[u] &= u_t - u_{xx} = f(x,t),\ 0 < x < \infty;\ t > 0, \\ u(x,0) &= g(x);\ u_x(0,t) = 1;\ u \to 0,\ x \to \infty. \end{aligned}$$

Derive the solution for the problem using the Green's function from first principles.
Hint: the free space Green's function is:

$$F(\xi, \tau, x, t) = \frac{H(t-\tau)}{(4\pi(t-\tau))^{1/2}} \exp\left(\frac{-(\xi - x)^2}{4(t-\tau)}\right)$$

where $H(t-\tau)$ is the unit step function.

3. [**16**] Consider the functional

$$F[u] = \int_0^1 \{(u'')^2 - ug(x)\}dx,$$

where $g(x)$ is a given function.
Determine, from first principles, the differential equation and natural boundary conditions that $u(x)$ must satisfy to minimize the functional.

4. [16] The region D is the triangle bounded by the lines, $y = \pm x/\sqrt{3}$ and $x = 3$, and

$$\begin{aligned} u_{xx} + u_{yy} + 2 &= 0 \text{ in } D, \\ u &= 0 \text{ on boundary of } D, \end{aligned} \qquad (1)$$

Describe how you would find approximate solutions to (1) using Galerkin, Rayleigh-Ritz and Kantorovich methods, specifically explaining the difference between the methods. Give details and set up integrals but **do not solve for constants**.

5. [17] (a) Write down an expression for the Rayleigh Quotient for the general Sturm-Liouville problem:

$$\begin{aligned} (p(x)u')' - q(x)u &= -\lambda r(x)u, \quad 1 \le x \le 2 \\ u(1) &= u(2) = 0 \end{aligned}$$

(b) For real α, solve

$$x^{-3}(x^5 u')' + \alpha u = 0.$$

(c) Give a rough numerical estimate of the lowest eigenvalue α_1 for

$$\begin{aligned} x^{-3}(x^5 u')' &= -\alpha u, \quad 1 \le x \le 2 \\ u(1) &= u(2) = 0. \end{aligned}$$

You can take $\ln 2 \sim 0.7$ and $\pi^2 \sim 10$.

(c) Obtain upper and lower bounds for the lowest eigenvalue, λ_1, for the eigenvalue problem:

$$\begin{aligned} x^{-3}((x^5 + 1)u')' &= -\lambda u, \quad 1 \le x \le 2 \\ u(1) &= u(2) = 0 \end{aligned}$$

by using different methods.

6. [17] (a) Find two term solutions for all three roots of

$$x^3 - (3 + \varepsilon)x - 2 + \varepsilon = 0,$$

for $\varepsilon \ll 1$.

(b) For $\varepsilon \ll 1$, find a one-term composite solution for

$$\begin{aligned} \varepsilon y'' + (1 + 3x)y' &= 1, \quad 0 \le x \le 1 \\ y(0) &= 2, \; y(1) = 1. \end{aligned}$$