

The University of British Columbia

Final Examination - December 5, 2012

Mathematics 400

Fall, 2012

Closed book examination

Time: 2.5 hours

Special Instructions:

No books, notes, or calculators are allowed.

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Part A		20
B1		5
B2		5
B3		5
B4		5
B5		5
B6		5
Total		50

Instructions:

- Do all ten questions in part A, each worth 2 points.
- Do all six questions in part B, each worth 5 points.
- total 50 marks.
- Justify all answers.
- Write your answers in booklets.

Part A: do all ten questions, each worth 2 marks

- A1.** [2marks] Write the D'Alembert form of the general solution to the wave equation for $u(x, t)$, $t \geq 0$ and all x :

$$u_{tt} = u_{xx}$$

- A2.** [2] Consider the following first order PDE problems. Decide whether each problem is linear, quasi-linear or fully nonlinear.

(a) $u(x, t): u_t + \sin(u)u_x = 0.$

(b) $u(x, y): x^2u_x + y^2u_y = e^{xy}.$

(c) $u(s, \theta): u_s u_\theta - \log u_s = 0.$

- A3.** [2] Determine whether the following PDE for $u(x, y)$ is elliptic, hyperbolic or parabolic:

$$u_{xx} + 3u_{xy} + 2u_{yy} + 8u_x + 3u_y = 0$$

- A4.** [2] Consider the following heat conduction problem for $u(x, t)$, $0 \leq x \leq 1$, $t \geq 0$:

$$u_t = u_{xx} + f(x, t)$$

$$u(0, t) = g_0(t)$$

$$u_x(1, t) = g_1(t)$$

$$u(x, 0) = u_0(x).$$

Draw a labelled diagram in the $x-t$ plane of the problem. Identify the initial conditions and boundary conditions on your diagram.

- A5.** [2] Consider the problem in A4 above with $f \equiv 0$ and $u_0 \equiv 0$ but with given nonzero functions $g_0(t)$ and $g_1(t)$. Describe briefly (a sentence or two) how you would go about deriving a series or transform solution to the problem.

- A6.** [2] Consider the following problem for $u(x, t)$, $0 \leq x \leq L$ and $t \geq 0$:

$$u_t + C u u_x + K u = Q$$

where L , C , K and Q are given constants. Scale x to unit length and scale u and T so that the resulting equation has as few dimensionless parameters as possible.

A7. [2] Consider the following system of equations for $u(x, t)$ and $v(x, t)$

$$\begin{aligned}u_t &= u_{xx} + f(u, v) \\v_t &= Dv_{xx} + g(u, v)\end{aligned}$$

where D is a positive constant and f and g are given functions with $f(0, 0) = 0$ and $g(0, 0) = 0$. Linearize the system about the solution $u \equiv 0$, $v \equiv 0$. Your answer will have coefficients that involve the partial derivatives of f and g .

A8. [2] Write the third order ODE for $u(t)$ below as a first order system.

$$\frac{d^3u}{dt^3} + \left(\frac{d^2u}{dt^2}\right)^2 - \sin(u) = 0.$$

A9. [2] Consider the heat conduction problem for $u(x, t)$, for $x \geq 0$ and $t \geq 0$:

$$\begin{aligned}u_t &= u_{xx} \\u_x(0, t) &= 1 \quad \text{for all } t \geq 0 \\u(x, 0) &= 0 \quad \text{for all } x \geq 0.\end{aligned}$$

Find $U(x, s)$, where U is the Laplace transform of u in t . **Do not** try to solve for $u(x, t)$.

A10. [2] Consider the following Sturm-Liouville problem for $\phi(x)$, $0 \leq x \leq 1$:

$$\phi'' = \begin{cases} \lambda\phi & x < 1/2 \\ 4\lambda\phi & x > 1/2 \end{cases}$$

with $\phi(0) = 0$ and $\phi(1) = 0$ and ϕ and ϕ' continuous at $x = 1/2$. It is known that all eigenvalues λ are negative. Write $\lambda = -\mu^2$ and find a function $f(\mu)$ such that the roots of f correspond to values μ that lead to eigenvalues of the problem. **Do not** solve for the μ values.

Part B: do all five questions, each worth 5 points

B1. Consider the following linear, first order Cauchy problem for $u(x, t)$, $t \geq 0$ and all x :

$$u_t + xu_x = \sin t$$

with initial data $u(x, 0) = e^{-x^2}$.

- (a) [2 marks] Solve for the characteristics $x(s, t)$ for this problem.
- (b) [2] Solve for $u(s, t)$ along these characteristics.
- (c) [1] Write the solution $u(x, t)$.

B2. Consider the traffic flow model we developed in class for $\rho(x, t)$:

$$\rho_t + [f(\rho)]_x = 0$$

where $f(\rho) = \rho(1 - \rho)$.

- (a) [2] Solve the Riemann Problem for this equation with $\rho_L = 0$ and $\rho_R = 1$.
 (b) [2] Solve the Riemann Problem for this equation with $\rho_L = 1$ and $\rho_R = 0$.
 (c) [1] In the $x - t$ plane, sketch the solution of the Cauchy problem for this equation with initial data

$$\rho(x, 0) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

B3. Consider the following wave equation problem for $u(x, t)$, $0 \leq x \leq 1$ and $t \geq 0$, with time harmonic forcing

$$u_{tt} = u_{xx} + e^x \cos t$$

with boundary conditions $u(0, t) = 0$ and $u_x(1, t) = 0$. Consider a time harmonic solution of the form

$$u(x, t) = \operatorname{Re} \{A(x)e^{it}\}$$

- (a) [2] Write the ODE that $A(x)$ satisfies.
 (b) [1] Write the boundary conditions that $A(x)$ satisfies.
 (c) [2] Solve for $A(x)$.
- B4.** Consider the heat conduction problem for $u(x, y, z, t)$, for $t \geq 0$, $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $0 \leq z \leq 1$:

$$u_t = \Delta u = u_{xx} + u_{yy} + u_{zz}$$

with homogeneous Dirichlet boundary conditions on all boundaries.

- (a) [3] Write the general solution to the problem as a series using the eigenvalues of the spatial operator. *Hint:* use sine series for x , y and z .
 (b) [2] What is the slowest decaying term in your series in part (a)? Your answer should be in the form

$$Ae^{-ct}v(x, y, z)$$

where A and c are constants you give values for and $v(x, y, z)$ is specified.

B5. Consider the following Sturm-Liouville problem for $\phi(x)$ with parameter $\epsilon \geq 0$:

$$\phi'' + (1 + \epsilon x)\phi' = \lambda\phi$$

with $\phi(0) = 0$ and $\phi(1) = 0$. It is known that all eigenvalues of this problem are negative.

- (a) [2] What is the orthogonality relationship for eigenfunctions of this problem?
- (b) [2] For $\epsilon = 0$ find the largest (least negative) eigenvalue λ and its corresponding eigenfunction.
- (c) [1] Discuss briefly (a sentence or two) how you could approach problem (b) with $\epsilon > 0$ but small.

B6. Use the Fourier transform to determine which of the following Cauchy problems for $u(x, t)$, $t \geq 0$ and all x , are well posed.

- (a) [2] $u_t = u_{xxx}$
- (b) [3] $u_{tt} + u_{txx} = u_{xx}$