Math 345 Final Exam (April 2009)

Last Name: ______________________________ **First name**: ______________________________

Student #: ______________________________ **Signature**: ______________________________

Circle your section #:

I have read and understood the instructions below:

Please sign:

Instructions:

1. No notes or books except those mentioned below are to be used in this exam.
2. You are allowed to bring a letter-sized formula sheet and a small-screen, non-graphic, non-programmable calculator.
3. Justify every answer whenever is necessary, and show your work. Unsupported answers will receive no credit.
4. You will be given 2.5 hrs to write this exam. Read over the exam before you begin. You are asked to stay in your seat during the last 5 minutes of the exam, until all exams are collected.
5. At the end of the hour you will be given the instruction "Put away all writing implements and remain seated." *Continuing to write after this instruction will be considered as cheating.*
6. **Academic dishonesty:** Exposing your paper to another student, copying material from another student, or representing your work as that of another student constitutes academic dishonesty. Cases of academic dishonesty may lead to a zero grade in the exam, a zero grade in the course, and other measures, such as suspension from this university.

Question	grade	value
1		24
2		16
3		16
4		20
5		24
Total		**100**

Question 1: **[24 marks]**

Consider the one-dimensional differential equation

$$\dot{x} = f(x, r) = x(r - 2x - x^2),$$

where the variable $x(t)$ and the parameter r both belong to the real line $\mathbf{R}$.

(a) Note that $x_s = 0$ is a fixed point for all r. (i) Use linear stability analysis to show that its stability changes at the bifurcation point $(x_s, r_c) = (0, 0)$. (ii) Determine the type of this bifurcation point (SN, TC, or PF). The normal form is not required.

(b) (i) Find all fixed points for all possible values of r (clearly state the conditions under which they exist). (ii) Plot the set $\mathcal{Z} = \{(x, r) : f(x, r) = 0\}$ (horizontal axis r, vertical axis x). (iii) Sketch all the possible qualitatively different one-dimensional phase portraits for different values of r. (No need to sketch the graph of $\dot{x}$ versus x).

(c) Based on results obtained in (b)(iii) or any other method(s), determine the stability of each fixed point in the plot obtained in (b)(ii). Sketch a bifurcation diagram (horizontal axis r, vertical axis x), showing all fixed points. Draw the stable branch(s) of fixed points with solid curves and the unstable branch(s) with dashed curves. (This diagram is identical in shape as that in (b)(ii) but is different only in separating the stable from the unstable fixed points).

(d) There is another bifurcation point in addition to that in (a). (i) Determine the values of x_s and r_c for this fixed point. (ii) Determine its type. (iii) Find its normal form.

Question 2: **[16 marks]**

Answer "True" or "False" to the statements below. **Put your answers in the boxes.**(4 mark each)

(a) A pitchfork bifurcation occurs in the equation $\dot{x} = x + \dfrac{rx}{1+x^2}$.

(b) The system $\dot{x} = x(1-x) - xy$ and $\dot{y} = y(1-y) - xy$ has no closed orbits.

(c) Consider a map $x_{n+1} = f(x_n)$. A period-2 cycle is characterized by a pair of values p and q such that $f(p) = q$ and $f(q) = p$, where both p and q are roots of $x = f^2(x) = f(f(x))$.

(d) Consider a map $x_{n+1} = f(x_n)$. A period-2 cycle (characterized by $f(p) = q$ and $f(q) = p$) is stable if the multiplier $\lambda = f'(p)f'(q)$ is positive valued.

Question 3: **[16 marks]**

These questions do not require lengthy calculations. **Put your answers in the boxes.** (4 marks each)

(a) For the tent map $x_{n+1} = f(x_n)$ where $f(x) = \begin{cases} rx, & 0 \le x \le 1/2 \\ r(1-x), & 1/2 \le x \le 1 \end{cases}$, where $r > 0$ is a parameter. Calculate the Liapunov exponent. Determine for what values of r the map has sensitive dependence on initial conditions.

(b) For the two-dimensional system in polar co-ordinates $\dot{r} = r(1-r)$, $\dot{\theta} = 1$ (where $r^2 = x^2 + y^2$ and $x = r\cos\theta$, $y = r\sin\theta$. Construct a Poincaré map at $\theta = 0$ (i.e. a map that generates a series of r values each time the trajectory goes through the line $\theta = 0$).

(c) For what values of a, b is the function $V(x, y) = ax^2 + by^2$ a Liapunov function of the system $\dot{x} = y - x^3$ and $\dot{y} = -x - y^3$?

(d) For the system $\dot{x} = xy$ and $\dot{y} = -x^2$. Find a quantity $E(x, y)$ that is conserved.

Question 4: **[20 marks]**

Consider the cubic map $x_{n+1} = f(x_n) = rx_n - x_n^3$ for the parameter r in $\mathbf{R}$.

(a) Find all the fixed points and determine for which values of r do they exist.

(b) Determine the stability of the fixed points found in (a) as a function of r.

(c) Find all the period-2 cycles. (Hint: Suppose $f(p) = q$ and $f(q) = p$. Show that p, q are roots of the equation $x(x^2 - r + 1)(x^2 - r - 1)(x^4 - rx^2 + 1) = 0$.)

(d) Determine the stability of the period-2 cycles as a function of r.

(e) Plot a partial bifurcation diagram, based on the information obtained for $-2 \leq r \leq 4$.

Question 5: **[24 marks]**

Odell (1980) developed the following predator-prey model

$$\begin{aligned}\dot{x} &= x[x(1-x)-y],\\ \dot{y} &= y(x-a),\end{aligned}$$

where $x \geq 0$ is the dimensionless population of the prey, $y \geq 0$ is the dimensionless population of the predator, and $a \geq 0$ is a control parameter.

(a) Sketch the nullclines in the first quadrant $x,\ y \geq 0$. Find all fixed points and mark each one by a small circle in the phase plane.

(b) Classify each fixed point and determine if each one is hyperbolic or non-hyperbolic.

(c) For $a = 1$, use linear stability analysis to determine the stability of each fixed point. Sketch the phase portrait for this case. Starting from a nonzero population size for both the predators and the preys, determine what happens to the predators after a long time.

(d) Determine the critical value of $a = a_c$ where a Hopf bifurcation occurs.

(e) Construct a trapping region in the first quadrant $x,\ y \geq 0$ within which there exists at least one closed orbit for $a < a_c$ (a_c is the value found in (d)). Sketch the shape of this region on the phase plane. [Hint: Show that on the interval $x \in (a, a+M)$ (for large enough $M > 0$) all flows on the line $y - M = -(x-a)$ are directed to $\dot{x} < 0$ and $\dot{y} > 0$ and that the ratio $\dot{x}/\dot{y} < -1$.]

(*More space for Question 5.*)

(*More space for Question 5.*)