Math 331, Section 201
Final Exam
April 17, 2007
Duration: 150 minutes

Name: ______________________ Student Number: ______________________

Do not open this test until instructed to do so! This exam should have 9 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed. Turn off any call phones, pagers, etc. that could make noise during the exam. You must remain in this room until you have finished the exam.

All your solutions must be written clearly and understandably. Use complete sentences and explain why your mathematical statements are relevant to the problem. You should always write enough to demonstrate that you're not just guessing the answer. Use the backs of the pages if necessary. You will find some of the questions quite easy; try to solve these first. Good luck!

Read these UBC rules governing examinations:

(i) Each candidate must be prepared to produce, upon request, a Library/AMS card for identification.
(ii) Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
(iii) No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
(iv) Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
- Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.
- Speaking or communicating with other candidates.
- Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

(v) Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Problem	**Out of**	**Score**
1	8	
2	8	
3	8	

Problem	**Out of**	**Score**
4	10	
5	8	
6	8	
Total	50	

Some helpful formulas

Remember: don't just apply formulas blindly in your solutions! Write enough to explain why any formula you use is relevant to the problem.

- If $\{a_n\} \overset{ops}{\longleftrightarrow} f(x)$ then, for $h \ge 1$:
  - $\{a_{n+h}\} \overset{ops}{\longleftrightarrow} \dfrac{f(x) - a_0 - a_1 x - \cdots - a_{h-1}x^{h-1}}{x^h}$
  - $\left\{ \displaystyle\sum_{n_1+\cdots+n_k=n} a_{n_1} a_{n_2} \cdots a_{n_k} \right\} \overset{ops}{\longleftrightarrow} f(x)^k$
- If $\{b_n\} \overset{egf}{\longleftrightarrow} g(x)$ then, for $h \ge 1$, $\{b_{n+h}\} \overset{egf}{\longleftrightarrow} D^h g(x)$.
- Some power series:

$$\sum_n \binom{\alpha}{n} x^n = (1+x)^\alpha \qquad \sum_n \binom{n+k}{n} x^n = \frac{1}{(1-x)^{k+1}}$$

$$\sum_n \frac{1}{n+1}\binom{2n}{n} x^n = \frac{1-\sqrt{1-4x}}{2x} \qquad \sum_n \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$$

$$\sum_{n\ge 0} \frac{k(2n+k-1)!}{n!(n+k)!} x^n = \left(\frac{1-\sqrt{1-4x}}{2x}\right)^k \qquad \sum_{n\ge 1} \frac{x^n}{n} = \log\frac{1}{1-x}$$

- Let $\mathcal{D}_1, \mathcal{D}_2, \dots$ be the decks of an exponential family, with d_n cards in $\mathcal{D}_n$; let $h(n,k)$ be the number of hands of weight n with k cards. Then with the definitions

$$\mathcal{D}(x) = \sum_{n\ge 1} d_n \frac{x^n}{n!} \qquad \text{and} \qquad \mathcal{H}(x,y) = \sum_{n\ge 0}\sum_{k\ge 0} h(n,k)\frac{x^n}{n!} y^k,$$

we have

$$\mathcal{H}(x,y) = e^{y\mathcal{D}(x)} \qquad \text{and} \qquad h(n,k) = \left[\frac{x^n}{n!}\right]\left\{\frac{\mathcal{D}(x)^k}{k!}\right\}.$$

- If $p(n,k)$ is the number of partitions of n into k parts, then

$$\sum_{n\ge 0}\sum_{k\ge 0} p(n,k) x^n y^k = \prod_{r\ge 1} \frac{1}{1-yx^r}.$$

- If $[x^n]F(x)$ counts the number (or proportion) of objects with a "score" of n, then the average μ and variance σ^2 of the scores are

$$\mu = \frac{F'(1)}{F(1)} \qquad \text{and} \qquad \sigma^2 = \left\{(\log F)' + (\log F)''\right\}\Big|_{x=1}.$$

• Let $N(\supseteq S)$ denote the number of objects with at least all the properties in S, and let E_t denote the number of objects with exactly t properties. Then:

$$N_r = \sum_{S\colon \#S=r} N(\supseteq S) \qquad\qquad E(x) = N(x-1)$$

$$N_r = \sum_{t\ge 0} \binom{t}{r} E_t \qquad\qquad E_t = \sum_{r\ge 0} (-1)^{r-t} \binom{r}{t} N_r$$

• Let C be a (strangely shaped) chessboard contained in an $n \times n$ board. If r_k is the number of ways of placing k non-attacking rooks on C, then the number of permutations of $\{1, 2, \dots, n\}$ that hit C in exactly j squares is

$$[x^j] \sum_k (n-k)! r_k (x-1)^k.$$

• Lagrange Inversion: If f, ϕ, and u are power series in t with $\phi(0) \ne 0$ and $u = t\phi(u)$, then

$$[t^n]\big\{f(u(t))\big\} = \frac{1}{n}[u^{n-1}]\big\{f'(u)\phi(u)^n\big\}.$$

• Let the principal part of a function f at a pole z_0 be

$$PP(f; z_0) = \sum_{j=1}^{r} \frac{a_{-j}}{(z-z_0)^j} = \sum_{n\ge 0} z^n \bigg\{ \sum_{j=1}^{r} \frac{(-1)^j a_{-j}}{z_0^{n+j}} \binom{n+j-1}{j-1} \bigg\}.$$

If the only poles of f inside the disk $\{z\colon |z| < R'\}$ are $z_1, \dots, z_s$, then

$$[z^n] f(z) = [z^n] \bigg\{ \sum_{k=1}^{s} PP(f; z_k) \bigg\} + O\bigg(\Big(\frac{1}{R'} + \varepsilon\Big)^n\bigg).$$

• Darboux's Theorem: Let $v(z) = \sum_{j\ge 0} v_j (1-z)^j$ be analytic in some disk $\{z\colon |z| < R\}$ with $R > 1$, and let β be a real number that isn't a nonnegative integer. Then for any $m \ge 0$,

$$[z^n]\big\{(1-z)^\beta v(z)\big\} = \sum_{j=0}^{m} v_j \binom{n-\beta-j-1}{n} + O\big(n^{-m-\beta-2}\big)$$

• Hayman's Theorem: If $f(z)$ is admissible, define $a(r) = rf'(r)/f(r)$ and $b(r) = ra'(r)$ and $a(r_n) = n$. Then

$$[z^n] f(z) \sim \frac{f(r_n)}{r_n^n \sqrt{2\pi b(r_n)}}.$$

If $p(z)$ is a polynomial with nonnegative coefficients and $[z^1]p(z) > 0$, then $e^{p(z)}$ is admissible.

1. **[8 pts]** A *derangement* of $\{1, 2, \ldots, n\}$ is a permutation π of $\{1, 2, \ldots, n\}$ such that $\pi(j) \neq j$ for every $1 \leq j \leq n$. Find, with proof, an explicit formula for the number of derangements of $\{1, 2, \ldots, n\}$.

2. **[8 pts]** Fix a positive real number k and a positive integer m. Determine the ordinary power series generating function $A(x) = \sum_{n\geq 0} a_n x^n$ for

$$a_n = \sum_r (-1)^r \binom{n - mr + r}{r} k^{n-rm}.$$

3.

(a) **[4 pts]** Prove "Stirling's formula":

$$n! \sim \frac{n^n}{e^n}\sqrt{2\pi n}$$

(b) **[4 pts]** By deriving an asymptotic formula for $\binom{3n}{n}$, find the radius of convergence R of the power series

$$\sum_{n\ge 0} \binom{3n}{n} x^n.$$

(Once you have the asymptotic formula for $\binom{3n}{n}$, you don't have to rigorously justify that your value of R is correct—just write it down.)

4. The *Bell number* $b(n)$ is the number of set partitions of the set $\{1, 2, \ldots, n\}$. Denote by $B(x) = \sum_{n \geq 0} b(n)x^n/n!$ the corresponding exponential generating function.

(a) **[5 pts]** Give a combinatorial argument to show that

$$b(n+1) = \sum_k \binom{n}{k} b(k). \qquad (*)$$

(Hint: sort the set partitions of $\{1, 2, \ldots, n+1\}$ into piles according to the number of elements in the part containing $n+1$.)

(b) **[5 pts]** Find the exponential generating functions of both sides of equation $(*)$, and solve the resulting equation to find $B(x)$.

5. Let $u(x) = \dfrac{1-\sqrt{1-4x}}{2x} - 1$.

   (a) **[3 pts]** Show that $u(x)$ is the unique function satisfying $u(0) = 0$ that is implicitly defined by $u = x(1+u)^2$.

   (b) **[5 pts]** For every positive integer k, prove the identity

$$\sum_{n\geq 0} \frac{k(2n+k-1)!}{n!(n+k)!} x^n = \left(\frac{1-\sqrt{1-4x}}{2x} \right)^k .$$

6. [**8 pts**] Find the exponential generating functions of

(a) $f_{\text{odd}}(n)$, the number of permutations of $\{1, \ldots, n\}$ with an odd number of cycles in their disjoint cycle decomposition; and

(b) $f_{\text{even}}(n)$, the number of permutations of $\{1, \ldots, n\}$ with an even number of cycles in their disjoint cycle decomposition.

From these exponential generating functions, conclude that $f_{\text{odd}}(n) = f_{\text{even}}(n)$ for all $n \geq 2$.