

Final Exam Duration: 150 minutes*This test has 6 questions on 12 pages, for a total of 75 points.*

- Do not turn this page over. You will have 150 minutes for the exam (between 15:30-18:00)
- This is a closed-book examination: no books, notes or electronic devices of any kind.
- You must justify all answers, regardless of the "operative word". Write in complete English sentences; proofs must be clear and concise.
- If you are using a result from the textbook, the lectures or the problem sets, state it properly.
- This exam is printed double-sided with the last two pages blank.

First Name: _____ Last Name: _____

Student-No: _____ Signature: _____

Question:	1	2	3	4	5	6	Total
Points:	18	6	18	10	18	5	75
Score:							

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (iii) purposely viewing the written papers of other examination candidates;
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

2 marks

1. (a) Define a *group homomorphism*.

4 marks

- (b) Let $f \in \text{Hom}(G, H)$. Show that $\text{Ker}(f)$ is a normal subgroup of G .

3 marks

- (c) Give an example of a group G and two subgroups $H, N < G$ such that N is normal in G but H isn't. No proof is required.

3 marks

(d) Are the permutations $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 6 & 2 & 7 & 8 & 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 8 & 6 & 1 & 2 & 7 & 3 \end{pmatrix}$ conjugate in S_8 ?

3 marks

(e) How many elements of order 25 are there in $C_{5^{100}}$? In $(C_5)^{100}$?

3 marks

(f) List the abelian groups of order 27

2. A *signed permutation matrix* is a square matrix w which has exactly one non-zero entry in each row and column, such entry being either 1 or -1 . Let $W_n \subset \text{GL}_n(\mathbb{R})$ be the set of $n \times n$ signed permutation matrices.

3 marks

(a) Show that W_n is a group.

3 marks

(b) Show that W_n is isomorphic to a semidirect product $S_n \ltimes (C_2)^n$.

3. Let X be a finite set, $G = S_X$.

7 marks

(a) For $A \subset X$ let $B = X \setminus A$ be its complement. Show that $\{\sigma \in G \mid \sigma(A) = A\} \simeq S_A \times S_B$.

5 marks

(b) Let $\kappa \in S_X$ be a k -cycle. Let A be its support, $B = \{x \in X \mid \kappa(x) = x\}$ its set of fixed points.

It is a fact that $Z_G(\kappa) \simeq \langle \kappa \rangle \times S_B$.

Use this fact to give a formula for the number of k -cycles in S_n .

6 marks

(c) Prove the fact from part (b).

4. Let G be a group, $A, B \subset G$.

5 marks

(a) Suppose that for all $a \in A, b \in B$ we have $aba^{-1} \in B$. Show that $\langle B \rangle$ is a normal subgroup of $\langle A \cup B \rangle$. You may use the fact that $\langle gBg^{-1} \rangle = g \langle B \rangle g^{-1}$.

5 marks

(b) Show that the image of A by the quotient map is a generating set of $\langle A \cup B \rangle / \langle B \rangle$

5. Let G be a group of order $351 = 27 \cdot 13$.

2 marks

(a) For $p = 3, 13$, what are the possible numbers of p -Sylow subgroups of G ?

4 marks

(b) Show that either $n_3(G) = 1$ or $n_{13}(G) = 1$, and conclude that G has the structure of a semidirect product.

6 marks

(c) Classify groups of order 351 with P_3 cyclic.

6 marks

(d) Classify groups of order 351 with $P_3 \simeq C_9 \times C_3$.

You may use that $(\mathbb{Z}/9\mathbb{Z})^\times$ is a cyclic group of order 6 and that $(\mathbb{Z}/27\mathbb{Z})^\times$ is a cyclic group of order 18.

5 marks

6. Let P be a finite p -group. Construct a finite group G with $n_p(G) > 1$ and such that the p -Sylow subgroups of G are all isomorphic to P . You may use the fact (due to Dirichlet) that there is a prime q such that $q \equiv 1(p)$.

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