Final Exam Duration: 150 minutes

*This test has **6 questions** on **12 pages**, for a total of 75 points.*

- Do not turn this page over. You will have 150 minutes for the exam (between 15:30-18:00)
- This is a closed-book examination: no books, notes or electronic devices of any kind.
- You must justify all answers, regardless of the "operative word". Write in complete English sentences; proofs must be clear and concise.
- If you are using a result from the textbook, the lectures or the problem sets, state it properly.
- This exam is printed double-sided with the last two pages blank.

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Signature: ______________________

Question:	1	2	3	4	5	6	Total
Points:	18	6	18	10	18	5	75
Score:							

2 marks 1. (a) Define a *group homomorphism*.

4 marks (b) Let $f \in \mathrm{Hom}(G, H)$. Show that $\mathrm{Ker}(f)$ is a normal subgroup of G.

3 marks (c) Give an example of a group G and two subgroups $H, N < G$ such that N is normal in G but H isn't. No proof is required.

$\boxed{\text{3 marks}}$ (d) Are the permutations $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 6 & 2 & 7 & 8 & 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 8 & 6 & 1 & 2 & 7 & 3 \end{pmatrix}$ conjugate in S_8?

$\boxed{\text{3 marks}}$ (e) How many elemenets of order 25 are there in $C_{5^{100}}$? In $(C_5)^{100}$?

$\boxed{\text{3 marks}}$ (f) List the abelian groups of order 27

2. A *signed permutation matrix* is a square matrix w which has exactly one non-zero entry in each row and column, such entry being either 1 or -1. Let $W_n \subset \mathrm{GL}_n(\mathbb{R})$ be the set of $n \times n$ signed permutation matrices.

   3 marks (a) Show that W_n is a group.

   3 marks (b) Show that W_n is isomorphic to a semidirect product $S_n \ltimes (C_2)^n$.

3. Let X be a finite set, $G = S_X$.

7 marks (a) For $A \subset X$ let $B = X \setminus A$ be its complement. Show that $\{\sigma \in G \mid \sigma(A) = A\} \simeq S_A \times S_B$.

5 marks (b) Let $\kappa \in S_X$ be a. Let A be its support, $B = \{x \in X \mid \kappa(x) = x\}$ its set of fixed points. It is a fact that $Z_G(\kappa) \simeq \langle \kappa \rangle \times S_B$.

Use this fact to give a formula for the number of k-cycles in S_n.

6 marks (c) Prove the fact from part (b).

4. Let G be a group, $A, B \subset G$.

5 marks (a) Suppose that for all $a \in A, b \in B$ we have $aba^{-1} \in B$. Show that $\langle B \rangle$ is a normal subgroup of $\langle A \cup B \rangle$. You may use the fact that $\langle gBg^{-1} \rangle = g \langle B \rangle g^{-1}$.

5 marks (b) Show that the image of A by the quotient map is a generating set of $\langle A \cup B \rangle / \langle B \rangle$

5. Let G be a group of order $351 = 27 \cdot 13$.

2 marks

(a) For $p = 3, 13$, what are the possible numbers of p-Sylow subgroups of G?

4 marks

(b) Show that either $n_3(G) = 1$ or $n_{13}(G) = 1$, and conclude that G has the structure of a semidirect product.

6 marks (c) Classify groups of order 351 with P_3 cyclic.

6 marks (d) Classify groups of order 351 with $P_3 \simeq C_9 \times C_3$.

You may use that $(\mathbb{Z}/9\mathbb{Z})^\times$ is a cyclic group of order 6 and that $(\mathbb{Z}/27\mathbb{Z})^\times$ is a cyclic group of order 18.

5 marks

6. Let P be a finite p-group. Construct a finite group G with $n_p(G) > 1$ and such that the p-Sylow subgroups of G are all isomorphic to P. You may use the fact (due to Dirichlet) that there is a prime q such that $q \equiv 1(p)$.

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