Marks

[9] **1.** Define

(a) $\int_a^b f(x)\, d\alpha(x)$

(b) a self–adjoint algebra of functions

(c) the Fourier series of a function

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[16] **2.** Give an example of each of the following, together with a brief explanation of your example. If an example does not exist, explain why not.

(a) A differentiable function which is not monotonic but whose derivative obeys $|f'(x)| \geq 1$.

(b) Two functions $f, \alpha : [0,1] \to \mathbb{R}$ with f continuous, but $f \notin \mathcal{R}(\alpha)$ on $[0,1]$.

(c) A continuous function $f : (-1,1) \to \mathbb{R}$ that cannot be uniformly approximated by a polynomial.

(d) A monotonically decreasing sequence of functions $f_n : [0,1] \to \mathbb{R}$ which converges pointwise, but not uniformly to zero.

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[15] **3.** Let f be a continuous function on $\mathbb{R}$. Suppose that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists? You must justify your conclusion.

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[15] **4.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is differentiable and that there is a number D such that

$$|f'(x)| \le D$$

for all $x \in [a, b]$. Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$, $T = \{t_1, \cdots, t_n\}$ be a choice for P and $S(P, T, f) = \sum_{i=1}^{n} f(t_i)[x_i - x_{-1}]$ be the corresponding Riemann sum. Prove that

$$\left| S(P, T, f) - \int_a^b f(x)\, dx \right| \le D\|P\|(b - a) \qquad \text{where } \|P\| = \max_{1 \le i \le n} [x_i - x_{i-1}]$$

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[15] **5.** Let $\{f_n : [0,1] \to \mathbb{R}\}_{n\in\mathbb{N}}$ be a sequence of continuous functions that obey $|f_n(y)| \le 1$ for all $n \in \mathbb{N}$ and all $y \in [0,1]$. Let $T : [0,1]\times[0,1] \to \mathbb{R}$ be continuous and define, for each $n \in \mathbb{N}$,

$$g_n(x) = \int_0^1 T(x,y)\, f_n(y)\ dy$$

Prove that the sequence $\{g_n\}_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.

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[15] **6.** (a) Let $H = \{ (x,y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 1 \}$. Prove that for any $\varepsilon > 0$ and any continuous function $f : H \to \mathbb{R}$ there exists a function $g(x,y)$ of the form

$$g(x,y) = \sum_{m=0}^{N} \sum_{n=0}^{N} a_{m,n} x^{2m} y^{2n} \qquad N \in \mathbb{Z},\ N \geq 0,\ a_{m,n} \in \mathbb{R}$$

such that

$$\sup_{(x,y)\in H} \big| f(x,y) - g(x,y) \big| < \varepsilon$$

(b) Does the result in (a) hold if H is replaced by the disk $\{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \}$?

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[15] **7.** The Legendre polynomials $P_n(x) : [-1,1] \to \mathbb{R}$, $n \in \mathbb{Z}$, $n \geq 0$, are polynomials obeying

(i) P_n is of degree n with the coefficient of x^n strictly greater than zero and

(ii) $\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{2}{2n+1} & \text{if } n = m \end{cases}$

Let $f : [-1,1] \to \mathbb{R}$ be continuous and set $a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx$. Prove that

(a) $\sum_{n=0}^{\infty} \frac{2}{2n+1}|a_n|^2 \leq \int_{-1}^{1} f(x)^2\,dx$ with equality if and only if $\sum_{n=0}^{N} a_n P_n(x)$ converges to f in the mean as $N \to \infty$.

(b) $\sum_{n=0}^{\infty} a_n P_n(x)$ converges in the mean to $f(x)$.

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Be sure that this examination has 12 pages including this cover

The University of British Columbia

Sessional Examinations - April 2006

Mathematics 321

Real Variables II

Closed book examination

Time: $2\frac{1}{2}$ hours

Name ______________________

Signature ______________________

Student Number ______________________

Instructor's Name ______________________

Section Number ______________________

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Total		100