Final Examination — December 8, 2016 Duration: 2.5 hours

This test has ***8 questions*** *on* ***10 pages****, for a total of 100 points.*

Dr. E. Perkins (Sec. 101) and Dr. I. Laba (Sec. 102)

- Continue on the back of the previous page if you run out of space, *with clear indication on the original page* that your solution is continued elsewhere.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Signature: ____

Question:	1	2	3	4	5	6	7	8	Total
Points:	12	10	7	14	12	21	14	10	100
Score:									

12 marks 1. Give examples of the following. Do not justify your examples.

(a) A countable dense subset, D, of $\mathbb{C}$.

(b) A countable subset, E, of the irrational numbers.

(c) A set $S \subset [0,1)$, $S \neq [0,1)$, which is relatively open in $[0,1)$ but is not open in $\mathbb{R}$.

$\boxed{\text{10 marks}}$ 2. Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. Assume $f : X \to Y$ and $g : Y \to Z$ are uniformly continuous. Prove that $g \circ f$ is also uniformly continuous.

$\boxed{\text{7 marks}}$ 3. Let F be an ordered field. Prove that F has no largest element, i.e. there is no $x \in F$ such that $y \leq x$ for all $y \in F$.

4. For each of the following give examples of metric spaces X and Y, and functions $f : X \to Y$ which satisfy the statement. You do not have to justify your examples.

$\boxed{\text{4 marks}}$ (a) f is continuous but not uniformly continuous.

$\boxed{\text{4 marks}}$ (b) f is continuous, X is not compact but $f(X)$ is compact.

$\boxed{\text{6 marks}}$ (c) $X = Y$, f is not continuous but the composition $f \circ f$ is continuous. You should give f and explicitly find $f \circ f$.

7 marks 5. (a) Show there is a sequence of real numbers for which the set of subsequential limits is $[0, 1]$.

5 marks (b) Prove there is no sequence of real numbers for which the set of subsequential limits is $(0, 1)$.

6. Decide if each of the following statements are True or False. Prove your answer.

$\boxed{\text{7 marks}}$ (a) $\sum_{n=1}^{\infty}(n^{1/n}-1)^n$ is convergent.

$\boxed{\text{7 marks}}$ (b) If a_n and b_n are real numbers such that $a_n \le b_n$ for all natural numbers n, then $\limsup_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n$.

7 marks

(c) If E is a subset of a metric space X, then the closure of the interior of E equals the closure of E.

7. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.

8 marks (a) If $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ are convergent, prove that $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

6 marks (b) Does the conclusion of (a) remain valid if we instead assume $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent? Prove or provide a counterexample.

$\boxed{\text{10 marks}}$ 8. Let X be a compact metric space. Prove that if a function $f : X \to \mathbb{R}$ is not bounded, then there is a point $x \in X$ such that $f(N_r(x))$ is not bounded for any neighbourhood $N_r(x)$ of X. (Note that we are **not** assuming that f is continuous.)