

Final Examination — December 8, 2016 **Duration: 2.5 hours***This test has 8 questions on 10 pages, for a total of 100 points.*

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- Continue on the back of the previous page if you run out of space, *with clear indication on the original page* that your solution is continued elsewhere.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: _____ Last Name: _____

Student-No: _____ Section: _____

Signature: _____

Question:	1	2	3	4	5	6	7	8	Total
Points:	12	10	7	14	12	21	14	10	100
Score:									

Student Conduct during Examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
- Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - speaking or communicating with other examination candidates, unless otherwise authorized;
 - purposely exposing written papers to the view of other examination candidates or imaging devices;
 - purposely viewing the written papers of other examination candidates;
 - using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)?(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

12 marks

1. Give examples of the following. Do not justify your examples.

(a) A countable dense subset, D , of \mathbb{C} .

(b) A countable subset, E , of the irrational numbers.

(c) A set $S \subset [0, 1)$, $S \neq [0, 1)$, which is relatively open in $[0, 1)$ but is not open in \mathbb{R} .

10 marks

2. Let (X, d_X) , (Y, d_Y) and (Z, d_Z) be metric spaces. Assume $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are uniformly continuous. Prove that $g \circ f$ is also uniformly continuous.

7 marks

3. Let F be an ordered field. Prove that F has no largest element, i.e. there is no $x \in F$ such that $y \leq x$ for all $y \in F$.

4. For each of the following give examples of metric spaces X and Y , and functions $f : X \rightarrow Y$ which satisfy the statement. You do not have to justify your examples.

4 marks

- (a) f is continuous but not uniformly continuous.

4 marks

- (b) f is continuous, X is not compact but $f(X)$ is compact.

6 marks

- (c) $X = Y$, f is not continuous but the composition $f \circ f$ is continuous. You should give f and explicitly find $f \circ f$.

7 marks

5. (a) Show there is a sequence of real numbers for which the set of subsequential limits is $[0, 1]$.

5 marks

- (b) Prove there is no sequence of real numbers for which the set of subsequential limits is $(0, 1)$.

6. Decide if each of the following statements are True or False. Prove your answer.

7 marks

(a) $\sum_{n=1}^{\infty} (n^{1/n} - 1)^n$ is convergent.

7 marks

(b) If a_n and b_n are real numbers such that $a_n \leq b_n$ for all natural numbers n , then $\limsup_{n \rightarrow \infty} a_n \leq \liminf_{n \rightarrow \infty} b_n$.

7 marks

- (c) If E is a subset of a metric space X , then the closure of the interior of E equals the closure of E .

7. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.

8 marks

(a) If $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ are convergent, prove that $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

6 marks

(b) Does the conclusion of (a) remain valid if we instead assume $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent? Prove or provide a counterexample.

10 marks

8. Let X be a compact metric space. Prove that if a function $f : X \rightarrow \mathbb{R}$ is not bounded, then there is a point $x \in X$ such that $f(N_r(x))$ is not bounded for any neighbourhood $N_r(x)$ of x . (Note that we are **not** assuming that f is continuous.)