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The University of British Columbia

MATH 317, Section 101

Final Exam – December 2016

Family Name \_\_\_\_\_ Given Name \_\_\_\_\_

Student Number \_\_\_\_\_ Signature \_\_\_\_\_

No notes or calculator allowed

### Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - i. speaking or communicating with other examination candidates, unless otherwise authorized;
  - ii. purposely exposing written papers to the view of other examination candidates or imaging devices;
  - iii. purposely viewing the written papers of other examination candidates;
  - iv. using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - v. using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Question	Points	Score
1	10	
2	10	
3	6	
4	6	
5	8	
6	8	
7	10	
8	12	
9	12	
10	18	
Total:	100	

1. A curve in  $\mathbb{R}^3$  is given by  $\vec{r}(t) = (t^2, t, t^3)$ .

(5 pt) (a) Find the parametric equations of the tangent line to the curve at the point  $P(1, -1, -1)$ .

(5 pt) (b) Find an equation for the osculating plane of the curve at the point  $Q(1, 1, 1)$ .

2. A curve in  $\mathbb{R}^3$  is given by  $\vec{\mathbf{r}}(t) = (\sin t - t \cos t)\hat{\mathbf{i}} + (\cos t + t \sin t)\hat{\mathbf{j}} + t^2\hat{\mathbf{k}}$ ,  $0 \leq t < \infty$ .

(6 pt) (a) Find the length of the curve  $\vec{\mathbf{r}}(t)$  from  $\vec{\mathbf{r}}(0) = (0, 1, 0)$  to  $\vec{\mathbf{r}}(\pi) = (\pi, -1, \pi^2)$ .

(4 pt) (b) Find the curvature of the curve at time  $t > 0$ .

- (6 pt) 3. Let  $\vec{\mathbf{F}}(x, y, z) = e^x \sin y \hat{\mathbf{i}} + [ae^x \cos y + bz] \hat{\mathbf{j}} + cx \hat{\mathbf{k}}$ . For which values of the constants  $a, b, c$  is  $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$  for all closed paths  $C$ ?

4. Let  $\vec{\mathbf{F}}(x, y) = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}$  be a smooth plane vector field defined for  $(x, y) \neq (0, 0)$ , and suppose  $Q_x = P_y$  for  $(x, y) \neq (0, 0)$ . In the following  $I_j = \int_{C_j} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$  for integer  $j$ , and all  $C_j$  are positively oriented circles. Suppose  $I_1 = \pi$  where  $C_1$  is the circle  $x^2 + y^2 = 1$ .

(2 pt) (a) Find  $I_2$  for  $C_2 : (x - 2)^2 + y^2 = 1$ . Explain briefly.

(2 pt) (b) Find  $I_3$  for  $C_3 : (x - 2)^2 + y^2 = 9$ . Explain briefly.

(2 pt) (c) Find  $I_4$  for  $C_4 : (x - 2)^2 + (y - 2)^2 = 9$ . Explain briefly.

- (8 pt) 5. Evaluate the line integral  $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} ds$  where  $\vec{\mathbf{F}}(x, y) = xy^2 \hat{\mathbf{i}} + ye^x \hat{\mathbf{j}}$ ,  $C$  is the boundary of the rectangle  $R: 0 \leq x \leq 3, -1 \leq y \leq 1$ , and  $\vec{\mathbf{n}}$  is the unit outer normal vector of  $C$ .

- (8 pt) 6. Find the work done by the force field  $\vec{\mathbf{F}}(x, y, z) = (x - y^2, y - z^2, z - x^2)$  on a particle that moves along the line segment from  $(0, 0, 1)$  to  $(2, 1, 0)$ .

- (10 pt) 7. Evaluate the surface integral  $\iint_S z^2 dS$  where  $S$  is the part of the cone  $x^2 + y^2 = 4z^2$  where  $0 \leq x \leq y$  and  $0 \leq z \leq 1$ .



(12 pt) 8. Let  $\vec{\mathbf{F}}$  be the vector field defined by

$$\vec{\mathbf{F}}(x, y, z) = (y^3 z + 2x) \hat{\mathbf{i}} + (3y - e^{\sin z}) \hat{\mathbf{j}} + (e^{x^2+y^2} + z) \hat{\mathbf{k}}.$$

Calculate the flux integral  $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$  where  $S$  is the boundary surface of the solid region

$$E: \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 2, \quad 0 \leq z \leq 2 + y$$

with outer normal.

9. Consider the vector field  $\vec{\mathbf{F}}(x, y, z) = z^2 \hat{\mathbf{i}} + x^2 \hat{\mathbf{j}} + y^2 \hat{\mathbf{k}}$  in  $\mathbb{R}^3$ .

- (6 pt) (a) Compute the line integral  $I_1 = \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ , where  $C_1$  is the curve consisting of three line segments,  $L_1$  from  $(2, 0, 0)$  to  $(0, 2, 0)$ , then  $L_2$  from  $(0, 2, 0)$  to  $(0, 0, 2)$ , finally  $L_3$  from  $(0, 0, 2)$  to  $(2, 0, 0)$ .

- (6 pt) (b) A simple closed curve  $C_2$  lies on the plane  $E : x + y + z = 2$ , enclosing a region  $R$  on the plane of area 3, and oriented in a counterclockwise direction as observed from the positive  $x$ -axis. Compute the line integral  $I_2 = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ .

(18 pt) 10. Circle True or False for each of the following statement. No justifications necessary.

Recall that  $f \in C^k$  means that all derivatives of  $f$  up to order  $k$  exist and are continuous.

- (a) True or False?  $\nabla \times (f\nabla f) = 0$  for all  $C^2$  scalar function  $f$  in  $\mathbb{R}^3$ .
- (b) True or False?  $\operatorname{div}(f\vec{\mathbf{F}}) = \nabla f \cdot \vec{\mathbf{F}} + f \operatorname{div} \vec{\mathbf{F}}$  for all  $C^1$  scalar function  $f$  and  $C^1$  vector field  $\vec{\mathbf{F}}$  in  $\mathbb{R}^3$ .
- (c) True or False? A smooth space curve  $C$  with constant curvature  $\kappa = 0$  must be a straight line.
- (d) True or False? A smooth space curve  $C$  with constant curvature  $\kappa \neq 0$  must be part of a circle of radius  $1/\kappa_0$ .
- (e) True or False? If  $f$  is any smooth function defined in  $\mathbb{R}^3$  and if  $C$  is any circle, then  $\int_C \nabla f \cdot d\vec{\mathbf{r}} = 0$ .
- (f) True or False? Suppose  $\vec{\mathbf{F}}$  is a smooth vector field in the region  $D$  which is  $\mathbb{R}^3$  without the origin. If  $\operatorname{div} \vec{\mathbf{F}} > 0$  through out  $D$ , then flux of  $\vec{\mathbf{F}}$  through the sphere of radius 5 with center at the origin is positive.
- (g) True or False? If  $\vec{\mathbf{F}}$  is a smooth vector field in  $\mathbb{R}^3$  and  $\operatorname{div} \vec{\mathbf{F}} = 0$  everywhere, then, for every sphere, the flux *out* of one hemisphere is equal to the flux *into* the opposite hemisphere.
- (h) True or False? Let  $\vec{\mathbf{F}}(x, y, z)$  be a continuously differentiable vector field which is defined for every  $(x, y, z)$ . Then,  $\iint_S \operatorname{curl} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = 0$  for any closed surface  $S$ .  
(A *closed surface* is a surface that is the boundary of a solid region.)
- (i) True or False? The parametrizations

$$\vec{\mathbf{r}}_1(t) = (t, t^2), \quad -\infty < t < \infty$$

and

$$\vec{\mathbf{r}}_2(t) = (t^2, t^4), \quad -\infty < t < \infty$$

describe exactly the same plane curve.

**Math 317 Calculus IV    Formulas for Final Exam**

1. For a curve  $\vec{\mathbf{r}}(t)$ ,  $s = \int_0^t |\vec{\mathbf{r}}'(\tau)| d\tau$ ,  $\frac{ds}{dt} = |\vec{\mathbf{r}}'|$ ,  $ds = |\vec{\mathbf{r}}'(t)| dt$
2.  $\mathbf{T} = \frac{\vec{\mathbf{r}}'}{|\vec{\mathbf{r}}'|}$ ,  $\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$ ,  $\mathbf{B} = \mathbf{T} \times \mathbf{N}$
3.  $\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{T}'|}{|\vec{\mathbf{r}}'|} = \frac{|\vec{\mathbf{r}}' \times \vec{\mathbf{r}}''|}{|\vec{\mathbf{r}}'|^3}$ ,  $\kappa \mathbf{N} = \frac{d\mathbf{T}}{ds}$
4. For  $y = f(x)$ ,  $\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$
5. Green's theorem:  $\int_C P dx + Q dy = \iint_D (Q_x - P_y) dA$
6. For a surface  $S$  given by  $\vec{\mathbf{r}}(u, v) : D \rightarrow \mathbb{R}^3$ , the surface area is  $\int_S dS = \iint_D |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| du dv$   
 $\int_S \rho dS = \iint_D \rho(\vec{\mathbf{r}}) |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| du dv$   
 $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} dS = \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}) \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) du dv$
7. For a graph  $S$  given by  $z = f(x, y)$ ,  $(x, y) \in D$ , the surface area is  
 $\int_S dS = \iint_D \sqrt{1 + f_x^2 + f_y^2} dx dy$   
 $\int_S \rho dS = \iint_D \rho(\vec{\mathbf{r}}) \sqrt{1 + f_x^2 + f_y^2} dx dy$   
 $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}) \cdot (-f_x, -f_y, 1) dx dy$
8. For a surface of revolution  $S : r = f(z)$ ,  $a \leq z \leq b$ , the surface area is  
 $\int_S dS = \int_a^b 2\pi f(z) \sqrt{1 + [f'(z)]^2} dz$
9. Stokes theorem: For a surface  $S$  with boundary curve  $C$ ,  $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S \text{curl } \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ .
10. Divergence theorem: For a solid region  $E$  with boundary surface  $S$ ,  
 $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_E \text{div } \vec{\mathbf{F}} dV$ .