Final Exam

December 11, 2013
19:00–21:30

No books. No notes. No calculators. No electronic devices of any kind.

Problem 1. (5 points)

(a) Consider the parametrized space curve

$$\vec{r}(t) = \langle t^2, t, t^3 \rangle \,.$$

Find an equation for the normal plane at the point $\langle 1, 1, 1 \rangle$.

(b) Find the curvature of the curve from (a) as a function of the parameter t.

Problem 2. (5 points)

(a) Let

$$\vec{r}(t) = \langle t^2, 3, \tfrac{1}{3}t^3 \rangle \,.$$

Find the unit tangent vector to this parametrized curve at $t = 1$, pointing in the direction of increasing t.

(b) Find the arc length of the curve from (a) between the points $(0, 3, 0)$ and $(1, 3, -\frac{1}{3})$.

Problem 3. (6 points)

(a) Consider the vector field

$$\vec{F}(x, y, z) = \langle z + e^y, xe^y - e^z \sin y, 1 + x + e^z \cos y \rangle \,.$$

Find the curl of $\vec{F}$. Is $\vec{F}$ conservative?

(b) Find the integral $\int_C \vec{F} \cdot d\vec{r}$ of the field $\vec{F}$ from (a) where C is the curve with parametrization

$$\vec{r}(t) = \langle t^2, \sin t, \cos^2 t \rangle \,,$$

where t ranges from 0 to π.

Problem 4. (6 points)

(a) Consider the vector field $\vec{F}(x, y, z) = \langle z^2, x^2, y^2 \rangle$ in $\mathbb{R}^3$. Compute the line integral $\oint_C \vec{F} \cdot d\vec{r}$, where C is the curve conisting of the three line segments, L_1 from $(2, 0, 0)$ to $(0, 2, 0)$, then L_2 from $(0, 2, 0)$ to $(0, 0, 2)$, and finally L_3 from $(0, 0, 2)$ to $(2, 0, 0)$.

(b) A simple closed curve C lies in the plane $x + y + z = 2$. The surface this curve C surrounds inside the plane $x + y + z = 2$ has area 3. The curve C is oriented in a counterclockwise direcction as observed from the positive x-axis. Compute the line integral $\oint_C \vec{F} \cdot d\vec{r}$, where F is as in (a).

Problem 5. (6 points)

(a) Find a parametrization of the surface S of the cone whose vertex is at the point $(0, 0, 3)$, and whose base is the circle $x^2 + y^2 = 4$ in the xy-plane. Only the cone surface belongs to S, not the base. Be careful to include the domain for the parameters.

(b) Find the z-coordinate of the centre of mass of the surface S from (a).

Problem 6. (6 points)

(a) Find an upward pointing unit normal vector to the surface $z = xy$ at the point $(1, 1, 1)$.

(b) Now consider the part of the surface $z = xy$, which lies within the cylinder $x^2 + y^2 = 9$ and call it S. Compute the upward flux of $\vec{F} = \langle y, x, 3 \rangle$ through S.

(c) Find the flux of $\vec{F} = \langle y, x, 3 \rangle$ through the cylindrical surface $x^2 + y^2 = 9$ in between $z = xy$ and $z = 10$. The orientation is outward, away from the z-axis.

Problem 7. (6 points)

(a) Find the divergence of the vector field $\vec{F} = \langle x + \sin y, z + y, z^2 \rangle$.

(b) Find the flux of $\vec{F}$ through the upper hemisphere $x^2 + y^2 + z^2 = 25$, $z \geq 0$, oriented in positive z-direction.

(c) Specify an oriented closed surface S, such that the flux $\iint_S \vec{F} \cdot d\vec{S}$ is equal to -9.

Problem 8. (10 points)
True or false? Put the answers in your exam booklet, please. No justifictions necessary.

1. $\vec{\nabla} \cdot (\vec{a} \times \vec{r}) = 0$, where $\vec{a}$ is a constant vector in $\mathbb{R}^3$, and $\vec{r}$ is the vector field $\vec{r} = \langle x, y, z \rangle$.
2. $\vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$, for all scalar fields f on $\mathbb{R}^3$ with continuous second partial derivatives.
3. $\operatorname{div}(f\,\vec{F}) = \overrightarrow{\operatorname{grad}}(f) \cdot \vec{F} + f\ \operatorname{div}\vec{F}$, for every vector field $\vec{F}$ in $\mathbb{R}^3$ with continuous partial derivatives, and every scalar function f in $\mathbb{R}^3$ with continuous partial derivatives.
4. Suppose $\vec{F}$ is a vector field with continuous partial derivatives in the region D, where D is $\mathbb{R}^3$ without the origin. If $\operatorname{div}\vec{F} > 0$ throughout D, then the flux of $\vec{F}$ through the sphere of radius 5 with center at the origin is positive.
5. Suppose $\vec{F}$ is a vector field with continuous partial derivatives in all of $\mathbb{R}^3$. Suppose further, that $\vec{\nabla} \times \vec{F}$ has positive z-component everywhere in $\mathbb{R}^3$. Then
$$\int_0^\pi \vec{F} \cdot \langle \cos\theta, \sin\theta, 0 \rangle \, d\theta > \int_0^\pi \vec{F} \cdot \langle \cos\theta, -\sin\theta, 0 \rangle \, d\theta \,.$$
6. If a vector field $\vec{F}$ is defined and has continuous partial derivatives everywhere in $\mathbb{R}^3$, and it satisfies $\operatorname{div}\vec{F} = 0$, everywhere, then, for every sphere, the flux *out of* one hemisphere is equal to the flux *into* the opposite hemisphere.
7. If $\vec{r}(t)$ is a twice continuously differentiable path in $\mathbb{R}^2$ with constant curvature κ, then $\vec{r}(t)$ parametrizes part of a circle of radius $1/\kappa$.
8. The vector field $\vec{F} = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$ is conservative in its domain, which is $\mathbb{R}^2$ without the origin.
9. If a vector field $\vec{F} = \langle P, Q \rangle$ in $\mathbb{R}^2$ has $Q = 0$ everywhere in $\mathbb{R}^2$, then the line integral $\oint \vec{F} \cdot d\vec{r}$ is zero, for every simple closed curve in $\mathbb{R}^2$.
10. If the acceleration and the speed of a moving particle in $\mathbb{R}^3$ are constant, then the motion is taking place along a spiral.