THE UNIVERSITY OF BRITISH COLUMBIA
Sessional Examinations – April 2010

MATHEMATICS 317

TIME: 2.5 hours
NO AIDS ARE PERMITTED. Each question is of equal value and is worth 10 points. Note that the maximum number of points is 70. A score of N/70 will be treated as N/55. Also note that this exam has **two** pages.

1. Consider the function $f(x, y) = xy$.
   (a) Explicitly determine the field lines (flow lines) of $\mathbf{F}(x, y) = \nabla f$.
   (b) Sketch the field lines of $\mathbf{F}$ and the level curves of f in the same diagram.

2. Suppose, in terms of the time parameter t, a particle moves along the path $\mathbf{r}(t) = (\sin t - t\cos t)\mathbf{i} + (\cos t + t\sin t)\mathbf{j} + t^2\mathbf{k}, \quad 1 \le t < \infty.$
   (a) Find the speed of the particle at time t.
   (b) Find the tangential component of acceleration at time t.
   (c) Find the normal component of acceleration at time t.
   (d) Find the curvature of the path at time t.

3. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be the position of a particle at time t. Suppose the motion of the particle satisfies the differential equation $\frac{d^2\mathbf{r}}{dt^2} = f(r)\mathbf{r}$ where $r = |\mathbf{r}|$.
   (a) Suppose $f(r)$ is an arbitrary function of r. Prove or disprove each of the following statements.
      (i) The motion of the particle is planar.
      (ii) The path of the particle sweeps out equal areas in equal times.
   (b) Find all forms of $f(r)$ for which the motion of the particle always lies on a straight line.
   (c) Give a specific form of $f(r)$ for which the motion of the particle could lie on an ellipse.

4. Let $\mathbf{F}(x, y, z) = \mathbf{r}/r^3$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$.
   (a) Find $\nabla \cdot \mathbf{F}$.
   (b) Find the flux of $\mathbf{F}$ outwards through the spherical surface $x^2 + y^2 + z^2 = a^2$.
   (c) Do the results of (a) and (b) contradict the divergence theorem? Explain your answer.
   (d) Let E be the solid region bounded by the surfaces $z^2 - x^2 - y^2 + 1 = 0$, $z = 1$ and $z = -1$. Let σ be the bounding surface of E. Determine the flux of $\mathbf{F}$ outwards through σ.
   (e) Let R be the solid region bounded by the surfaces $z^2 - x^2 - y^2 + 4y - 3 = 0$, $z = 1$ and $z = -1$. Let Σ be the bounding surface of R. Determine the flux of $\mathbf{F}$ outwards through Σ.

5. Let σ_1 be the open surface given by $z = 1 - x^2 - y^2, \quad z \ge 0$. Let σ_2 be the open surface given by $z = x^2 + y^2 - 1, \quad z \le 0$. Let σ_3 be the planar surface given by $z = 0, \quad x^2 + y^2 \le 1$. Let $\mathbf{F} = [a(y^2 + z^2) + bxz]\mathbf{i} + [c(x^2 + z^2) + dyz]\mathbf{j} + x^2\mathbf{k}$ where $a, b, c,$ and d are constants.
   (a) Find the flux of $\mathbf{F}$ upwards across σ_1.
   (b) Find all values of the constants $a, b, c,$ and d so that the flux of $\mathbf{F}$ outwards across the closed surface $\sigma_1 \cup \sigma_3$ is zero.
   (c) Find all values of the constants $a, b, c,$ and d so that the flux of $\mathbf{F}$ outwards across the closed surface $\sigma_1 \cup \sigma_2$ is zero.

6. Let C be the curve defined by the intersection of the surfaces $z = x + y$ and $z = x^2 + y^2$.
   (a) Show that C is a simple closed curve.
   (b) Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where
      (i) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + 3e^z\mathbf{k}$.
      (ii) $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j} + 3e^z\mathbf{k}$.

7. Let S be the surface $z = 2 + x^2 - 3y^2$ and $\mathbf{F}(x, y, z) = (xz + axy^2)\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$. Consider the points $P_1 = (1,1,0)$ and $P_2 = (0,0,2)$ on the surface S. Find a value of the constant a so that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for *any* two curves C_1 and C_2 on the surface S from P_1 to P_2.