Math 313 Topics in Number Theory—Final Exam

April 22, 2013

1. (10 points) Find all solutions (x, y) to the equation $x^2 - 17y^2 = \pm 1$ for which $x, y > 0$. In other words, find a fundamental solution (x, y) and explain how all other solutions with $x, y > 0$ can be generated from it.

2. (10 points) In the ring $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$, show that $\sqrt{-3} \mid a + b\frac{1+\sqrt{-3}}{2}$ if and only if $a \equiv b \mod 3$.

3. (10 points) Suppose that a and b are both kth-power residues mod n. Show that ab is also a kth-power residue. Hint: this is the same as the same proof for quadratic residues mod a prime.

4. (20 points) Let p be a prime, and suppose $x^2 + xy - 3y^2 = mp$ where m is some integer and $1 < |m| < p$. Find some $a \equiv x \mod m$ and $b \equiv y \mod m$ such that $a^2 + ab - 3b^2 = km$ and $|k| < |m|$. Explain briefly why this implies that $\mathbb{Z}[\frac{1+\sqrt{13}}{2}]$ has unique factorization.

5. (10 points) Suppose that $d \equiv 5 \mod 8$. Show that there are no elements of norm 2 in $\mathbb{Z}[\sqrt{d}]$; deduce that the element 2 is irreducible in $\mathbb{Z}[\sqrt{d}]$. Show then that $2 \mid d^2 - d = (d + \sqrt{d})(d - \sqrt{d})$; conclude that 2 is not prime and therefore that there is no unique factorization in $\mathbb{Z}[\sqrt{d}]$. This is similar to but not the same as the homework question asking about the case in which d has a prime factor $p \equiv 5 \mod 8$; this question covers some additional cases, such as $d = 21$.

6. (25 points) Show that the the only integer solutions to the equation $x^2 + 11 = y^3$ are $x = \pm 4, y = 3$ and $x = \pm 58, y = 15$; you may quote without proof the fact that there is unique factorization in $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$. Hint: when I did this problem, in the final step I wrote $(a + b\sqrt{-11})^3 = x + \sqrt{-11}$ but allowed a and b be half-integers, and then I argued why a priori the only possible values for b are $\pm 1, \pm\frac{1}{2}$.

7. (15 points) Let $\left(\frac{d}{p}\right) = -1$ with p an odd prime. Show that if $x + y\sqrt{d}$ has norm divisible by p then $p \mid x + y\sqrt{d}$. Conclude that if $p \mid (x_1 + y_1\sqrt{d})(x_2 + y_2\sqrt{d})$ then $p \mid x_1 + y_1\sqrt{d}$ or $p \mid x_2 + y_2\sqrt{d}$.