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The University of British Columbia

MATH 308, Section 101, Instructor Tai-Peng Tsai

Final Exam – December 2010

Family Name ______________________ **Given Name** ______________________

Student Number ______________________ **Signature** ______________________

No notes nor calculators.

Rules Governing Formal Examinations:

1. Each candidate must be prepared to produce, upon request, a UBCcard for identification;

2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions;

3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination;

4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action;
(a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
(b) Speaking or communicating with other candidates;
(c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received;

5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers, and must not take any examination material from the examination room without permission of the invigilator; and

6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

problem	max	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	15	
7.	15	
8.	10	
9.	10	
total	100	

(10 points) 1. Find the two pairs of focuses and directrices of the ellipse $x^2 + 4y^2 + 2x = 0$.

(10 points) 2. Let $a > 0$, $b > 0$, and F and G be the parabolas $y^2 = 4a(x + a)$ and $y^2 = 4b(-x + b)$. The origin is the focus of both parabolas. Suppose F meets G at P above the x-axis. Use the reflection property of parabolas to prove that F and G cross at P at right angle.

3. Classify the conics in $\mathbb{R}^2$ with the following equations. You do not need to find their axes.

(5 points) (a) $3x^2 - 8xy + 2y^2 - 2x + 4y - 16 = 0$

(5 points) (b) $x^2 + 8xy + 16y^2 - x + 8y - 12 = 0$

(10 points) 4. On a triangle ABC, D and D' are points on side BC so that $\angle BAD = \angle D'AC$, E and E' are points on side CA so that $\angle CBE = \angle E'BA$, and F and F' are points on sides AB so that $\angle ACF = \angle F'CB$. Suppose that AD, BE, CF are concurrent. Show that AD', BE', CF' are also concurrent. Hint: You may use $\dfrac{BD}{DC} = \dfrac{AB \sin \angle BAD}{AC \sin \angle DAC}$.

(10 points) 5. Find an affine transformation which maps the ellipse $x^2 + 4y^2 + 2x = 0$ to the unit circle. (It is not unique, but you only need to find one.)

(5 points) 6. (a) Find the affine transformation which maps the points $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ to the points $\begin{pmatrix} 8 \\ 13 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, respectively.

(5 points) (b) Find the projective transformation which maps the Points $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$ and $[1, 1, 1]$ to the Points $[2, 1, 0]$, $[1, 0, -1]$, $[0, 3, -1]$ and $[3, -1, 2]$.

(5 points) (c) Find the Point of intersection of the Lines $L_1 : x + 2y + 5z = 0$ and $L_2 : 3x - y + z = 0$.

7. Let $A = [2, 1, 1]$, $B = [-1, 1, -1]$, $C = [1, 2, 0]$ and $D = [-1, 4, -2]$ be four points in $\mathbb{RP}^2$.

(5 points) (a) Verify that A, B, C, D are collinear and find the equation of the Line.

(5 points) (b) Find the cross ratio $(ABCD)$.

(5 points) (c) Find a point X in the same line so that $(ABCX) = -1$.

(10 points) 8. Find the imagies of the circles $|z| = 1$ and $|z| = 2$ under the Möbius transformation $t(z) = \dfrac{z-2}{z+i}$.

(10 points) 9. Let C be a circle with radius r and centered at O. For any ordinary point $P \neq O$, let P' be the inversion of P in C. Then the line p through P' and perpendicular to OPP' is called the *polar* of P for the circle C.

Show that, if the polar of P passes through a point Q, then the polar of Q passes through P.

Some Formulas

	eccentricity	**standard eq**	**focus**	**directrix**	**tangent at** (x_1, y_1)
ellipse	$0 \le e < 1$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$(\pm ae, 0)$	$x = \pm a/e$	$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$
hyperbola	$e > 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$(\pm ae, 0)$	$x = \pm a/e$	$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1$
parabola	$e = 1$	$y^2 = 4ax$	$(a, 0)$	$x = -a$	$y_1 y = 2a(x + x_1)$

Above $b = a\sqrt{|1 - e^2|}$

1. ellipsoid $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
2. hyperboloids of one and two sheets, elliptic cone $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, -1, 0$
3. elliptic paraboloid and hyperbolic paraboloid $\quad z = \frac{x^2}{a^2} \pm \frac{y^2}{b^2}$
4. The affine transformation of $\mathbb{R}^2$ mapping $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ to $P = \left[\begin{smallmatrix}p_1\\p_2\end{smallmatrix}\right]$, $Q = \left[\begin{smallmatrix}q_1\\q_2\end{smallmatrix}\right]$, and $R = \left[\begin{smallmatrix}r_1\\r_2\end{smallmatrix}\right]$ is
$$t(\vec{x}) = \vec{P} + \mathbf{A}\vec{x}, \quad \mathbf{A} = (\vec{PQ}|\vec{PR}) = \begin{pmatrix} q_1 - p_1 & r_1 - p_1 \\ q_2 - p_2 & r_2 - p_2 \end{pmatrix}.$$
5. Affine transformations preserve collinearity, coincidence, parallel lines, and signed ratio of lengths along the same direction.
6. Ceva's and Menelaus' theorems: $P \in BC$, $Q \in CA$, $R \in AB$, $\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = \pm 1$ implies coincidence or collinearity.
7. The projective transformation of $\mathbb{RP}^2$ mapping $[1,0,0]$, $[0,1,0]$, $[0,0,1]$ and $[1,1,1]$ to $P = [p_1, p_2, p_3]$, $Q = [q_1, q_2, q_3]$, $R = [r_1, r_2, r_2]$ and $S = [s_1, s_2, s_3]$ is
$$t([\vec{x}]) = [\mathbf{A}\vec{x}], \quad \mathbf{A} = [uP^\intercal | vQ^\intercal | wR^\intercal], \quad \text{where} \quad [P^\intercal|Q^\intercal|R^\intercal]\left[\begin{smallmatrix}u\\v\\w\end{smallmatrix}\right] = S^\intercal.$$
Projective transformations preserve collinearity, coincidence, and cross ratio.
8. The Line passing Points P and Q has equation $\det(X^\intercal|P^\intercal|Q^\intercal) = 0$.
9. If $A = [a], B = [b], C = [c], D = [d]$ are 4 collinear points with $c = \alpha a + \beta b$ and $d = \gamma a + \delta b$, then $(ABCD) = (\beta/\alpha)/(\delta/\gamma) = (AC/CB)/(AD/DB)$. If $(ABCD) = k$, then $(BACD) = (ABDC) = 1/k$ and $(ACBD) = (DBCA) = 1 - k$.
10. Inversion in the unit circle: $(x, y) \mapsto (\frac{x}{r^2}, \frac{y}{r^2})$ where $r = \sqrt{x^2 + y^2}$. Inversion in the circle $|z - c| = R$ is $z \mapsto \frac{R^2}{\bar{z} - \bar{c}} + c$.
11. The stereographic projection $\pi : \mathbb{S}^2 \to \mathbb{C}$ satisfies $\pi : (X, Y, Z) \mapsto \frac{X + iY}{1 - Z}$ with $\pi^{-1} : x + iy \mapsto \frac{1}{x^2 + y^2 + 1}(2x, 2y, x^2 + y^2 - 1)$.