University of British Columbia
Math 307, Section 101 (Froese)
Final Exam, December 2015

Name (print): ______________________________

Student ID Number: ____________________ **Signature:** ______________________

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise authorized;
  (b) purposely exposing written papers to the view of other candidates or imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Question	Points	Score
1	15	
2	14	
3	12	
4	15	
5	16	
6	10	
7	18	
Total:	100	

Additional Instructions:

- No notes, books or calculators are allowed. A MATLAB/Octave formula sheet is provided on the last page.
- Read the questions carefully and make sure you provide all the information that is asked for in the question.
- Show all your work. Correct answers without explanation or accompanying work could receive no credit.
- Answer the questions in the space provided. Continue on the back of the page if necessary.

1. Let

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \sqrt{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

(a) (3 points) For what real values of a (if any) is $\|A\| = 4$?

(b) (3 points) For what real values of a (if any) is $\text{cond}(A) = 4$?

(c) (3 points) Compute the streching ratio $\|B\mathbf{x}\|/\|\mathbf{x}\|$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(d) (3 points) Use the calculation in the previous part to determine $\|B\|$ and $\text{cond}(B)$.

(e) (3 points) Suppose C is a 3×3 matrix with $\text{cond}(C) = 10$. If $C \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $C \begin{bmatrix} 1 \\ 1+a \\ 1 \end{bmatrix} = \begin{bmatrix} 1.1 \\ 0 \\ 0 \end{bmatrix}$, what are the possible values of a?

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2. Let (x_i, y_i), $i = 1, \ldots 4$ be four points in the plane with $x_1 < x_2 < x_3 < x_4$.

   (a) (3 points) If the polynomial $p(x) = a_1 x^3 + a_2 x^2 + a_3 x + a_4$ interpolates the four points, then the coefficient vector $\mathbf{a} = [a_1, a_2, a_3, a_4]^T$ satisfies an equation of the form $A\mathbf{a} = \mathbf{d}$. Write down A and $\mathbf{d}$.

(b) (3 points) If the polynomial $p(x) = b_1x^4 + b_2x^3 + b_3x^2 + b_4x + b_5$ interpolates the four points, and also satisfies $p'(x_4) = 0$, then the coefficient vector $\mathbf{b} = [b_1, b_2, b_3, b_4, b_5]^T$ satisfies an equation of the form $B\mathbf{b} = \mathbf{d}$. Write down B and $\mathbf{d}$.

(c) (4 points) If the polynomial $p(x) = c_1x^2 + c_2x + c_3$ interpolates the four points, then the coefficient vector $\mathbf{c} = [c_1, c_2, c_3]^T$ satisfies an equation of the form $C\mathbf{a} = \mathbf{e}$. Write down C and $\mathbf{e}$.

(d) (4 points) For each of the equations in parts (a) (b) and (c) say whether you expect there to be a solution. For the case(s) where you do not expect a solution, write down the least squares equation. Do these have a solution? Give a reason. What quantity is minimized when the least squares equation is satisfied?

3. Consider the chemical system consisting of the species H_2SO_4, HSO_4^-, SO_4^{--} and H^+. In addition to the species H, S and O, we also regard the charge as a species q. Thus the formula matrix is

$$A = \begin{array}{c} \\ H \\ S \\ O \\ q \end{array} \begin{array}{c} \begin{array}{cccc} H_2SO_4 & HSO_4^- & SO_4^{--} & H^+ \end{array} \\ \left(\begin{array}{cccc} 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 4 & 4 & 4 & 0 \\ 0 & -1 & -2 & 1 \end{array} \right) \end{array}.$$

After defining `A` in MATLAB/Octave, we compute

```
>rref(A)
ans =

   1   0  -1   1
   0   1   2  -1
   0   0   0   0
   0   0   0   0
```

```
>rref(A')
ans =

   1   0   0   1
   0   1   4  -2
   0   0   0   0
   0   0   0   0
```

(a) (4 points) Write down a basis for $N(A)$ and for $N(A^T)$.

(b) (4 points) Write down the possible reactions for this system.

(c) (4 points) If a sample contains 350 atoms of H, 200 atoms of S and 800 atoms of O, what is the total charge q? (Hint: what subspace contains $[350, 200, 800, q]^T$?)

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4. (a) (4 points) Write down two conditions that must be satisfied in order for the functions $\phi_n(t)$, for $t \in [0, 1]$ and $n \in \mathbb{Z}$ to form an orthonormal set. Do the functions $e^{2\pi i n t}$ satisfy these conditions?

   (b) (3 points) Do the functions $\phi_n(t) = (1/\sqrt{2})e^{2\pi i n t}$ on the larger interval $t \in [0, 2]$, form an orthonormal set? Can we expand any (sufficiently nice) function defined for $t \in [0, 2]$ in a series $\sum_{n=-\infty}^{\infty} c_n \phi_n(t)$

(c) (4 points) Find the coefficients c_n in the Fourier series

$$e^{i\pi t} = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int}$$

where $t \in [0, 1]$.

(d) (4 points) What does Parseval's formula say for the series in part (c)?

5. Let

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}.$$

(a) (4 points) Is A diagonalizable? Give a reason.

(b) (4 points) Schur's lemma states that there is a unitary matrix U and an upper triangular matrix T such that $A = UTU^*$. Find U and T.

(c) (4 points) What are T^2 and T^3? Guess a formula for T^n for any positive integer n, and use it to compute A^n.

(d) (4 points) Write down a formula for the nth term, x_n, in the sequence defined by the recurrence relation $x_0 = a$, $x_1 = b$ and $x_{n+1} = 2x_n - x_{n-1}$ for $n \geq 1$.

6. Suppose A is an 5×5 real symmetric matrix defined in MATLAB/Octave. The commands

```
> x=rand(5,1)
>for k=1:50 y=(A-3*eye(5))\x; x=y/norm(y) end
```

yield output ending in

```
x =                         x =                         x =


   0.12692                    -0.12692                     0.12692
  -0.32566                     0.32566                    -0.32566
   0.80572                    -0.80572                     0.80572
  -0.11046                     0.11045                    -0.11046
  -0.46524                     0.46524                    -0.46524
```

If the eigenvalues of A are $0, 0.5, 1.5, 2.5, 4$ what output would you get for

(a) (4 points) `dot(x,A*x)`

(b) (3 points) `dot(x,x)`

(c) (3 points) `dot(y,x)`

7. (a) (4 points) Let $P = [p_{i,j}]$ be an $n \times n$ matrix. What does it mean to say that P is a stochastic matrix? If P is stochastic, what can you say about the eigenvalues and eigenvectors? What additional information do you have about the eigenvalues and eigenvectors if P^k has all positive entries for some k?

(b) (4 points) Draw the 4 site internet represented by the stochastic matrix

$$P = \begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1 & 0 & 0 \\ 1/3 & 0 & 1/2 & 1 \end{bmatrix}.$$

(c) (3 points) By examining this internet or otherwise, find an eigenvector of P with eigenvalue 1.

(d) (3 points) Given that the eigenvalues of P satisfy $\lambda_1 = 1, |\lambda_2| = 0.65034, |\lambda_3| = |\lambda_4| = 0.50624$, what is $\lim_{k\to\infty} P^k \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$?

(e) (4 points) If we add damping with a damping factor $\alpha = 1/2$ what is the new stochastic matrix S? What can you say about $\lim_{k\to\infty} S^k \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$?

`pi`	π
`i`	$\sqrt{-1}$
`x = 3`	define variable **x** to be 3
`x = [1 2 3]`	set **x** to the 1×3 row vector $(1, 2, 3)$
`x = [1; 2; 3]`	set **x** to the 3×1 vector $(1, 2, 3)$
`A = [1 2; 3 4]`	set A to the 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
`x(2) = 7`	change x_2 to 7
`A(2,1) = 0`	change A_{21} to 0
`3*x`	multiply each element of **x** by 3
`x+3`	add 3 to each element of **x**
`x+y`	add **x** and **y** element by element
`A*x`	product of matrix A and column vector **x**
`A*B`	product of two matrices A and B
`x.*y`	element-wise product of vectors **x** and **y**
`A^3`	for a square matrix A, raise to third power
`cos(A)`	cosine of every element of A
`sin(A)`	sine of every element of A
`x'`	transpose of vector **x**
`A'`	transpose of vector A
`A(2:12,4)`	the submatrix of A consisting of the second to twelfth rows of the fourth column
`A(2:12,4:5)`	the submatrix of A consisting of the second to twelfth rows of the fourth and fifth columns
`A(2:12,:)`	the submatrix of A consisting of the second to twelfth rows of all columns
`A([1:4,6],:)`	the submatrix of A consisting of the first to fourth rows and sixth row
`[A B; C D]`	creates the matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C, D are block matrices (blocks must have compatible sizes)
`rand(12,4)`	12×4 matrix with uniform random numbers in $[0, 1)$
`zeros(12,4)`	12×4 matrix of zeroes
`ones(12,4)`	12×4 matrix of ones
`eye(12)`	12×12 identity matrix
`eye(12,4)`	12×4 matrix whose first 4 rows are the 4×4 identity
`linspace(1.2,4.7,100)`	row vector of 100 equally spaced numbers from 1.2 to 4.7
`diag(x)`	matrix whose diagonal is the entries of **x** (other elements are zero)
`diag(x,n)`	matrix whose diagonal is the entries of **x** on diagonal n (other elements are zero)
`sum(x)`	sum of the elements of **x**

`A\b`	returns the solution **x** to $A\mathbf{x} = \mathbf{b}$
`A^(-1)`	returns the inverse of A
`rref(A)`	returns the reduced row echelon form of A
`det(A)`	returns the determinant of A
`norm(A)`	returns the (operator) norm of A
`cond(A)`	returns the condition number of A
`length(A)`	returns the larger of the number of rows and number of columns of A
`norm(x)`	returns the norm (length) of a vector **x**
`vander(x)`	returns the Vandermonde matrix for the points of **x**
`polyval(a,x)`	returns the values of the polynomial $a_1x^{n-1} + a_2x^{n-2} + \ldots a_n$ at the points of **x**
`[Q R] = qr(A,0)`	returns the matrices Q and R in the QR factorization of A
`nextpow2(N)`	calculates the next power of 2 of N
`fft(f,N)`	FFT transform of the vector **f** using N points (pads **f** with zeros if it has fewer than N elements)
`polyval(A)`	returns the coefficients of the characteristic polynomial of A
`roots(a)`	returns the solutions to $a_1x^{n-1} + a_2x^{n-2} + \ldots a_n = 0$
`[V D] = eig(A)`	returns the matrix V whose columns are normalized eigenvectors of A and the diagonal matrix D of corresponding eigenvalues
`plot(x,y,'bo')`	plots the points of **y** against the points of **x** using blue dots
`plot(x,y,'r-')`	plots the points of **y** against the points of **x** using red lines
`semilogy(x,y,'bo')`	plots **y** against **x** using a logarithmic scale for **y**
`axis([-0.1 1.1 -3 5])`	changes the axes of the plot to be from -0.1 to 1.1 for the x-axis and -3 to 5 for the y-axis
`hold on`	puts any new plots on top of the existing plot
`hold off`	any new plot commands replace the existing plot (this is the default)
`plot3(x,y,z,'bo')`	plots the points of **z** against the points of **x** and **y** using blue dots
`for k=1:10 ... end`	for loop taking `k` from `1` to `10` and performing the commands ... for each