The University of British Columbia

16 December 2013, 7pm-9:30pm

Final Exam for MATH 307 Section 101 Winter 2013

Calculators and one double-sided letter-paper sized formula sheet are permitted
Time: 2 1/2 hours

Last Name ______________________________ **First** __________________

Signature _________________

Student Number ___________________

Special Instructions:

Except for one double-sided hand-written letter-paper sized formula sheet, no memory aids are allowed. No communication devices. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page. **Where boxes are provided for answers, put your final answers in them.**

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		30
2		10
3		10
4		30
Total		80

[30] **1.** Short answer:

(a) What four conditions does a cubic spline satisfy at each data point (x_i, y_i)?

(b) Write down the linear system required to compute the quadratic function passing through the points $(3, 5), (-1, 5), (1, 3)$.

(c) Calculate the inner products and norms for the following:

(i) The vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 5 \\ -1 \end{bmatrix}$,

(ii) The vectors $\begin{bmatrix} 1+i \\ 3-i \end{bmatrix}$ and $\begin{bmatrix} 2-2i \\ 4+3i \end{bmatrix}$,

(iii) The functions e^{3it} and e^{-it} for $-\pi \le t \le \pi$.

(d) Write down the integral you need to do to compute the Fourier coefficient c_1 of the basis function $e_1(t) = e^{2\pi i t}$ for the function $f(t) = \sin(2\pi t)$, $0 \le t \le 1$. What is the value of coefficient c_2?

1. continued

(e) Suppose $B = A(A^TA)^{-1}A^T$ for some matrix A, $x = [1,1,1,1]^T$, and

$$Bx = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Compute Bx and B^Tx for $x = [1,1,0,0]^T$.

(f) Draw a graph showing the feasible set of the linear program

$$\begin{cases} \text{minimize} & 3x + 4y \\ \text{subject to} & x + y \leq 10, \quad x \leq y, \quad x, y \geq 0 \end{cases}$$

1. continued

(g) Match each problem (P1-P4) to the most appropriate matrix factorization (F1-F5).

P1 $Ax = b$, where the problem is overdetermined

P2 $Ax = b$, where the problem is non-singular or underdetermined

P3 $Ax = \lambda x$, A Hermetian

P4 $Ax = \lambda x$, A square

F1 $A = LU$

F2 $A = C^T C$

F3 $A = SDS^{-1}$

F4 $A = QR$

F5 $D = A - B^T C^{-1} B$

F6 $A = QDQ^T$

F7 $A = UDV^T$

F8 $A = UDU^*$

[10] **2.** Consider the recurrance relation

$$x_{n+2} = (3/2)x_{n+1} - (1/2)x_n \tag{1}$$

(a) Write the transition matrix for this recurrance relation.

(b) What is $\lim_{n\to\infty} x_n$:

(i) when $x_0 = 1, x_1 = 1,$

(ii) when $x_0 = 2, x_1 = 1,$

(iii) when $x_0 = 3, x_1 = 2,$

(iv) when $x_0 = p + 2q, x_1 = p + q.$

[10] **3.** Suppose, in addition to standard matrix-vector operations such as multiplication, transpose, addition, subtraction, multplication by a scalar, and so forth, you have a computer language implementing the following functions:

`x=A\b;` solves the *non-singular* linear system $Ax = b$

`[Q,R]=qr(A);` computes the QR-factorization of matrix

(a) Write down how you would use this computer language to solve a full-rank overdetermined system of equations by the QR-factorization. Annotate each line with a comment describing what the line does, and the line's time complexity.

(b) Give two reasons to prefer solving least-squares problems by the QR-factorization over solving the normal equations.

[30] **4.** Consider the node-edge incidence matrix

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \tag{2}$$

and its row-reduced forms:

$$\mathrm{rref}(D) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathrm{rref}(D^T) = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

(a) Compute the four subspaces for D: the nullspace, columnspace (also called range), rowspace and left-nullspace. What is the rank if D?

(b) Draw the graph represented by D, numbering the edges and vertices corresponding with the row and column indices of D.

(c) Set $b = [-3, 2, -1, 2, 1, -1]^T$. Show that $Dx = b$ has at least one solution.

4. continued

(d) Find the general solution to $Dx = b$ for b in part(c).

(e) Give one possible collection of independent loops for the graph given by D.

(f) Suppose the edges have conductances $c = [1, 2, 2, 1, 3, 1]$. What is the graph Laplacian L? What is the nullspace of L?

(g) It can be shown that when $c \geq 0$, all eigenvalues of the graph Laplacian are non-negative. What can you say about the quadratic form $x^T Lx$? What value(s) of x give $x^T Lx = 0$?