The University of British Columbia

Final Examination - December 13, 2012

Mathematics 307/101

Closed book examination — Time: 2.5 hours

Last Name ____________ **First** ________ **Signature** ____________

Student Number ____________

Special Instructions:

No books, notes, or calculators are allowed. A MATLAB/Octave formula sheet is provided on the last page.

1		20
2		18
3		16
4		16
5		14
6		16
Total		100

[20] **1**. In this question we will work with polynomials of degree 3 written

$$p(x) = a_1x^3 + a_2x^2 + a_3x + a_4$$

(a) [4 pts]

The coefficient vector $\mathbf{a} = [a_1, a_2, a_3, a_4]^T$ satisfies an equation of the form $A\mathbf{a} = \mathbf{0}$ when the slopes of $p(x)$ at $x = 0$ and $x = 2$ are zero. Write down the matrix A.

(b) [4 pts]

Show that $\dim(N(A)) = 2$ and find a basis $\mathbf{a}_1, \mathbf{a}_2$ for $N(A)$.

(c) [4 pts]

The coefficient vector $\mathbf{a} = [a_1, a_2, a_3, a_4]^T$ satisfies an equation of the form $B\mathbf{a} = \mathbf{b}$ when the graph of $p(x)$ passes through the points $(0, 1)$, $(1, 2)$ and $(2, 2)$. Write down the matrix B and the vector $\mathbf{b}$.

(d) [4 pts]

Using the equation in (c) find the equation $C\mathbf{s} = \mathbf{c}$ satisfied by $\mathbf{s} = [s_1, s_2]^T$ if $p(x)$

(i) has coefficient vector $\mathbf{a} = s_1\mathbf{a}_1 + s_2\mathbf{a}_2$ (and therefore has zero slopes at $x = 0$ and $x = 2$).

(ii) passes through the points $(0, 1)$, $(1, 2)$ and $(2, 2)$.

Does this equation have a solution? Give a reason.

(e) [4 pts]

Write down the MATLAB/Octave code that plots the points $(0, 1)$, $(1, 2)$ and $(2, 2)$ and the polynomial $p(x)$ that

(i) has has zero slopes at $x = 0$ and $x = 2$.

(ii) comes closest in the least squares sense to passing through the points $(0, 1)$, $(1, 2)$ and $(2, 2)$.

[18] **2**. The boundary value problem

$$\begin{aligned} &f''(x) + xf(x) = 1, \quad 0 < x < 1 \\ &f(0) = 1, \quad f'(1) = 1 \end{aligned}$$

can be approximated by an $(N+1) \times (N+1)$ system of linear equations of the form

$$(L + (\Delta x)^2 Q)\mathbf{F} = \mathbf{b}$$

(a) [10 pts]

Write down L, Q, $\mathbf{b}$ and Δx when $N = 4$.

(b) [8 pts]

How would you use MATLAB/Octave to compute approximations to $f(1/2)$ and $f'(1/2)$? Assume that N has been defined and write code that uses this value of N.

[16] **3**. Let $S = \left\{[x_1, x_2, x_3]^T : x_1 + x_2 + x_3 = 0\right\}$ be the subspace of vectors in $\mathbb{R}^3$ whose components sum to zero.

(a) [2 pts]

Find a matrix A so that S is the null space of A, i.e., $S = N(A)$.

(b) [3 pts]

Write down a basis for S.

(c) [3 pts]

Find a matrix B so that $R(B) = S$

(d) [4 pts]

Write down the MATLAB/Octave code that

(i) computes the projection matrix P that projects onto S and

(ii) computes the vector in S that is closest to $[0, 1, 0]^T$.

(e) [4 pts]

Let $Q = I - P$. What kind of matrix is Q? What are $N(Q)$ and $R(Q)$?

[16] **4**. Consider the Fourier series

$$t^2 - t = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t}$$

for $0 \le t \le 1$.

(a) [3 pts]

What is the definition of the inner product $\langle f, g\rangle$ for two complex valued functions $f(t)$ and $g(t)$ defined for $0 \le t \le 1$?

(b) [3 pts]

The coefficient c_n can be written as an inner product $c_n = \langle f, g\rangle$. What are f and g?

(c) [3 pts]

Compute the coefficient c_0.

(d) [3 pts]

Given that $c_n = \frac{1}{2\pi^2 n^2}$ for $n \neq 0$, use Parseval's formula to find the value of the infinite sum $\sum_{n=1}^{\infty} \frac{1}{n^4}$. Calculate a numerical expression - you do not need to simplify your answer.

(e) [4 pts]

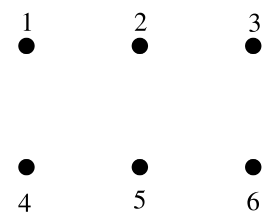
What is the value of $\int_0^1 \cos(2\pi t)(t^2 - t)dt$? (Hint: use that $\cos(2\pi t) = (1/2)(e^{2\pi i t} + e^{-2\pi i t})$)

[14] **5**. Consider an internet with six sites. The stochastic matrix describing the behaviour of a web surfer (with damping factor zero) is

$$P = \begin{bmatrix} 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

(a) [4 pts]

Draw the internet links correponding to the matrix P on the following diagram.



(b) [4 pts]

If we begin equal probabilities of being at each site, what are the probabilities of being a each site after one step? What is the probability of being at site 6 after two steps?

(c) [4 pts]

Introduce a damping factor of $1/2$ so that the surfer now moves according to the stochastic matrix $(1/2)P + (1/2)Q$ where Q is the stochastic matrix describing a surfer picking any site at random (including the current site). Beginning equal probabilities of being at each site, what are the probabilities of being a each site after one step?

(d) [2 pts]

Each set below contains the absolute values of the eigenvalues of either P or $(1/2)P + (1/2)Q$. Check your guess for which list is which below and give a reason for your choice.

$\{1.00, 0.42, 0.19, 0.19, 0.10, 0.25\}$ are the values: for P $\square$, for $(1/2)P + (1/2)Q$ $\square$.

$\{1.00, 0.85, 0.38, 0.38, 0.21, 0.50\}$ are the values: for P $\square$, for $(1/2)P + (1/2)Q$ $\square$.

[16] **6**. Suppose that A is a matrix with singular value decomposition $A = U\Sigma V^*$ where

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

(a) [4 pts] Check all the boxes that apply.

U and V are: hermitian □, real symmetric □, unitary □, orthogonal □.

Σ is: hermitian □, real symmetric □, unitary □, orthogonal □.

A^*A and AA^* are: hermitian □, real symmetric □, unitary □, orthogonal □.

A is: hermitian □, real symmetric □, unitary □, orthogonal □.

(b) [2 pts]

What are the eigenvalues and eigenvectors of A^TA? (Note: $A^T = A^*$ since A is real.)

(c) [2 pts]

What is the matrix norm $\|A\|$? Give a reason

(d) [2 pts]

Write down the singular value decomposition for A^{-1}.

(e) [2 pts]

What are the matrix norm $\|A^{-1}\|$ and the condition number of A?

(f) [4 pts]

Let

$$\hat{A} = U \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^*,$$

where U and V are the matrices given above. Then $\hat{A}$ is a matrix with a non-trivial null space. What is a basis for $N(\hat{A})$? What is the norm $\|\hat{A} - A\|$?

The End

`pi`	π
`i`	$\sqrt{-1}$
`x = 3`	define variable $\mathbf{x}$ to be 3
`x = [1 2 3]`	set $\mathbf{x}$ to the 1×3 row vector $(1, 2, 3)$
`x = [1; 2; 3]`	set $\mathbf{x}$ to the 3×1 vector $(1, 2, 3)$
`A = [1 2; 3 4]`	set A to the 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
`x(2) = 7`	change x_2 to 7
`A(2,1) = 0`	change A_{21} to 0
`3*x`	multiply each element of $\mathbf{x}$ by 3
`x+3`	add 3 to each element of $\mathbf{x}$
`x+y`	add $\mathbf{x}$ and $\mathbf{y}$ element by element
`A*x`	product of matrix A and column vector $\mathbf{x}$
`A*B`	product of two matrices A and B
`x.*y`	element-wise product of vectors $\mathbf{x}$ and $\mathbf{y}$
`A^3`	for a square matrix A, raise to third power
`cos(A)`	cosine of every element of A
`sin(A)`	sine of every element of A
`x'`	transpose of vector $\mathbf{x}$
`A'`	transpose of vector A
`A(2:12,4)`	the submatrix of A consisting of the second to twelfth rows of the fourth column
`A(2:12,4:5)`	the submatrix of A consisting of the second to twelfth rows of the fourth and fifth columns
`A(2:12,:)`	the submatrix of A consisting of the second to twelfth rows of all columns
`A([1:4,6],:)`	the submatrix of A consisting of the first to fourth rows and sixth row
`[A B; C D]`	creates the matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C, D are block matrices (blocks must have compatible sizes)
`rand(12,4)`	12×4 matrix with uniform random numbers in $[0, 1)$
`zeros(12,4)`	12×4 matrix of zeroes
`ones(12,4)`	12×4 matrix of ones
`eye(12)`	12×12 identity matrix
`eye(12,4)`	12×4 matrix whose first 4 rows are the 4×4 identity
`linspace(1.2,4.7,100)`	row vector of 100 equally spaced numbers from 1.2 to 4.7
`diag(x)`	matrix whose diagonal is the entries of $\mathbf{x}$ (other elements are zero)
`diag(x,n)`	matrix whose diagonal is the entries of $\mathbf{x}$ on diagonal n (other elements are zero)
`sum(x)`	sum of the elements of $\mathbf{x}$

`A\b`	returns the solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$
`A^(-1)`	returns the inverse of A
`rref(A)`	returns the reduced row echelon form of A
`det(A)`	returns the determinant of A
`norm(A)`	returns the (operator) norm of A
`cond(A)`	returns the condition number of A
`length(A)`	returns the larger of the number of rows and number of columns of A
`norm(x)`	returns the norm (length) of a vector $\mathbf{x}$
`vander(x)`	returns the Vandermonde matrix for the points of $\mathbf{x}$
`polyval(a,x)`	returns the values of the polynomial $a_1x^{n-1} + a_2x^{n-2} + \ldots a_n$ at the points of $\mathbf{x}$
`[Q R] = qr(A,0)`	returns the matrices Q and R in the QR factorization of A
`nextpow2(N)`	calculates the next power of 2 of N
`fft(f,N)`	FFT transform of the vector $\mathbf{f}$ using N points (pads $\mathbf{f}$ with zeros if it has fewer than N elements)
`polyval(A)`	returns the coefficients of the characteristic polynomial of A
`roots(a)`	returns the solutions to $a_1x^{n-1} + a_2x^{n-2} + \ldots a_n = 0$
`[V D] = eig(A)`	returns the matrix V whose columns are normalized eigenvectors of A and the diagonal matrix D of corresponding eigenvalues
`plot(x,y,'bo')`	plots the points of $\mathbf{y}$ against the points of $\mathbf{x}$ using blue dots
`plot(x,y,'r-')`	plots the points of $\mathbf{y}$ against the points of $\mathbf{x}$ using red lines
`semilogy(x,y,'bo')`	plots $\mathbf{y}$ against $\mathbf{x}$ using a logarithmic scale for $\mathbf{y}$
`axis([-0.1 1.1 -3 5])`	changes the axes of the plot to be from -0.1 to 1.1 for the x-axis and -3 to 5 for the y-axis
`hold on`	puts any new plots on top of the existing plot
`hold off`	any new plot commands replace the existing plot (this is the default)
`plot3(x,y,z,'bo')`	plots the points of $\mathbf{z}$ against the points of $\mathbf{x}$ and $\mathbf{y}$ using blue dots
`for k=1:10 ... end`	for loop taking `k` from `1` to `10` and performing the commands ... for each