

The University of British Columbia

Final Examination - December 13, 2012

Mathematics 307/101

Closed book examination

Time: 2.5 hours

Last Name \_\_\_\_\_ First \_\_\_\_\_ Signature \_\_\_\_\_

Student Number \_\_\_\_\_

Special Instructions:

No books, notes, or calculators are allowed. A MATLAB/Octave formula sheet is provided on the last page.

**Rules governing examinations**

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (a) speaking or communicating with other candidates, unless otherwise authorized;
  - (b) purposely exposing written papers to the view of other candidates or imaging devices;
  - (c) purposely viewing the written papers of other candidates;
  - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		20
2		18
3		16
4		16
5		14
6		16
Total		100

[20] 1. In this question we will work with polynomials of degree 3 written

$$p(x) = a_1x^3 + a_2x^2 + a_3x + a_4$$

(a) [4 pts]

The coefficient vector  $\mathbf{a} = [a_1, a_2, a_3, a_4]^T$  satisfies an equation of the form  $A\mathbf{a} = \mathbf{0}$  when the slopes of  $p(x)$  at  $x = 0$  and  $x = 2$  are zero. Write down the matrix  $A$ .

(b) [4 pts]

Show that  $\dim(N(A)) = 2$  and find a basis  $\mathbf{a}_1, \mathbf{a}_2$  for  $N(A)$ .

(c) [4 pts]

The coefficient vector  $\mathbf{a} = [a_1, a_2, a_3, a_4]^T$  satisfies an equation of the form  $B\mathbf{a} = \mathbf{b}$  when the graph of  $p(x)$  passes through the points  $(0, 1)$ ,  $(1, 2)$  and  $(2, 2)$ . Write down the matrix  $B$  and the vector  $\mathbf{b}$ .

(d) [4 pts]

Using the equation in (c) find the equation  $C\mathbf{s} = \mathbf{c}$  satisfied by  $\mathbf{s} = [s_1, s_2]^T$  if  $p(x)$

- (i) has coefficient vector  $\mathbf{a} = s_1\mathbf{a}_1 + s_2\mathbf{a}_2$  (and therefore has zero slopes at  $x = 0$  and  $x = 2$ ).
- (ii) passes through the points  $(0, 1)$ ,  $(1, 2)$  and  $(2, 2)$ .

Does this equation have a solution? Give a reason.

(e) [4 pts]

Write down the MATLAB/Octave code that plots the points  $(0, 1)$ ,  $(1, 2)$  and  $(2, 2)$  and the polynomial  $p(x)$  that

- (i) has zero slopes at  $x = 0$  and  $x = 2$ .
- (ii) comes closest in the least squares sense to passing through the points  $(0, 1)$ ,  $(1, 2)$  and  $(2, 2)$ .

[18] 2. The boundary value problem

$$\begin{aligned}f''(x) + xf(x) &= 1, & 0 < x < 1 \\f(0) &= 1, & f'(1) = 1\end{aligned}$$

can be approximated by an  $(N + 1) \times (N + 1)$  system of linear equations of the form

$$(L + (\Delta x)^2 Q)\mathbf{F} = \mathbf{b}$$

(a) [10 pts]

Write down  $L$ ,  $Q$ ,  $\mathbf{b}$  and  $\Delta x$  when  $N = 4$ .

(b) [8 pts]

How would you use MATLAB/Octave to compute approximations to  $f(1/2)$  and  $f'(1/2)$ ? Assume that  $N$  has been defined and write code that uses this value of  $N$ .

[16] **3.** Let  $S = \{[x_1, x_2, x_3]^T : x_1 + x_2 + x_3 = 0\}$  be the subspace of vectors in  $\mathbb{R}^3$  whose components sum to zero.

(a) [2 pts]

Find a matrix  $A$  so that  $S$  is the null space of  $A$ , i.e.,  $S = N(A)$ .

(b) [3 pts]

Write down a basis for  $S$ .

(c) [3 pts]

Find a matrix  $B$  so that  $R(B) = S$

(d) [4 pts]

Write down the MATLAB/Octave code that

- (i) computes the projection matrix  $P$  that projects onto  $S$  and
- (ii) computes the vector in  $S$  that is closest to  $[0, 1, 0]^T$ .

(e) [4 pts]

Let  $Q = I - P$ . What kind of matrix is  $Q$ ? What are  $N(Q)$  and  $R(Q)$ ?

[16] 4. Consider the Fourier series

$$t^2 - t = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t}$$

for  $0 \leq t \leq 1$ .

(a) [3 pts]

What is the definition of the inner product  $\langle f, g \rangle$  for two complex valued functions  $f(t)$  and  $g(t)$  defined for  $0 \leq t \leq 1$ ?

(b) [3 pts]

The coefficient  $c_n$  can be written as an inner product  $c_n = \langle f, g \rangle$ . What are  $f$  and  $g$ ?

(c) [3 pts]

Compute the coefficient  $c_0$ .

(d) [3 pts]

Given that  $c_n = \frac{1}{2\pi^2 n^2}$  for  $n \neq 0$ , use Parseval's formula to find the value of the infinite sum  $\sum_{n=1}^{\infty} \frac{1}{n^4}$ . Calculate a numerical expression - you do not need to simplify your answer.

(e) [4 pts]

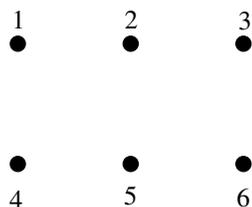
What is the value of  $\int_0^1 \cos(2\pi t)(t^2 - t)dt$ ? (Hint: use that  $\cos(2\pi t) = (1/2)(e^{2\pi it} + e^{-2\pi it})$ )

[14] 5. Consider an internet with six sites. The stochastic matrix describing the behaviour of a web surfer (with damping factor zero) is

$$P = \begin{bmatrix} 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

(a) [4 pts]

Draw the internet links corresponding to the matrix  $P$  on the following diagram.



(b) [4 pts]

If we begin equal probabilities of being at each site, what are the probabilities of being at each site after one step? What is the probability of being at site 6 after two steps?

(c) [4 pts]

Introduce a damping factor of  $1/2$  so that the surfer now moves according to the stochastic matrix  $(1/2)P + (1/2)Q$  where  $Q$  is the stochastic matrix describing a surfer picking any site at random (including the current site). Beginning equal probabilities of being at each site, what are the probabilities of being at each site after one step?

(d) [2 pts]

Each set below contains the absolute values of the eigenvalues of either  $P$  or  $(1/2)P + (1/2)Q$ . Check your guess for which list is which below and give a reason for your choice.

$\{1.00, 0.42, 0.19, 0.19, 0.10, 0.25\}$  are the values: for  $P$  , for  $(1/2)P + (1/2)Q$  .

$\{1.00, 0.85, 0.38, 0.38, 0.21, 0.50\}$  are the values: for  $P$  , for  $(1/2)P + (1/2)Q$  .

[16] 6. Suppose that  $A$  is a matrix with singular value decomposition  $A = U\Sigma V^*$  where

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

(a) [4 pts] Check all the boxes that apply.

$U$  and  $V$  are: hermitian , real symmetric , unitary , orthogonal .

$\Sigma$  is: hermitian , real symmetric , unitary , orthogonal .

$A^*A$  and  $AA^*$  are: hermitian , real symmetric , unitary , orthogonal .

$A$  is: hermitian , real symmetric , unitary , orthogonal .

(b) [2 pts]

What are the eigenvalues and eigenvectors of  $A^T A$ ? (Note:  $A^T = A^*$  since  $A$  is real.)

(c) [2 pts]

What is the matrix norm  $\|A\|$ ? Give a reason

(d) [2 pts]

Write down the singular value decomposition for  $A^{-1}$ .

(e) [2 pts]

What are the matrix norm  $\|A^{-1}\|$  and the condition number of  $A$ ?

(f) [4 pts]

Let

$$\hat{A} = U \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^*,$$

where  $U$  and  $V$  are the matrices given above. Then  $\hat{A}$  is a matrix with a non-trivial null space. What is a basis for  $N(\hat{A})$ ? What is the norm  $\|\hat{A} - A\|$ ?

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pi      pi
1      1
x = 3
x = [1 2 3]
x = [1; 2; 3]
A = [1 2; 3 4]
x(2) = 7
x(2,1) = 0
3*x
x**3
x+y
A*x
A*B
x.*y
A^-3
cos(A)
sin(A)
x'
A'
A(2:12,4)
A(2:12,4:5)
A(2:12,:)
A([1:4,6],:)
[A B; C D]

rand(12,4)
zeros(12,4)
ones(12,4)
eye(12)
eye(12,4)
linspace(1,2,4,7,100)
diag(x)
diag(x,n)
sum(x)

pi
sqrt(-1)
define variable x to be 3
set x to the 1 x 3 row vector (1,2,3)
set x to the 3 x 1 vector (1,2,3)
set A to the 2 x 2 matrix [ 1 2
                          3 4 ]
change x2 to 7
multiply each element of x by 3
add 3 to each element of x
add x and y element by element
product of matrix A and column vector x
product of two matrices A and B
element-wise product of vectors x and y
for a square matrix A, raise to third power
cosine of every element of A
sine of every element of A
transpose of vector x
transpose of vector A
the submatrix of A consisting of the second to twelfth rows of the fourth column
the submatrix of A consisting of the second to twelfth rows of the fourth and
fifth columns
the submatrix of A consisting of the second to twelfth rows of all columns
the submatrix of A consisting of the first to fourth rows and sixth row
creates the matrix [ A B
                    C D ] where A, B, C, D are block matrices (blocks must
have compatible sizes)
12 x 4 matrix with uniform random numbers in [0,1)
12 x 4 matrix of zeros
12 x 4 matrix of ones
12 x 12 identity matrix
12 x 4 matrix whose first 4 rows are the 4 x 4 identity
row vector of 100 equally spaced numbers from 1.2 to 4.7
matrix whose diagonal is the entries of x (other elements are zero)
matrix whose diagonal is the entries of x on diagonal n (other elements are
zero)
sum of the elements of x

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A\b
A^(-1)
rref(A)
det(A)
norm(A)
cond(A)
length(A)
norm(x)
vander(x)
polyval(a,x)
[Q R] = qr(A,0)
nextpow2(N)
fft(f,N)
polyval(A)
roots(a)
[V D] = eig(A)
plot(x,y,'bo')
plot(x,y,'r-')
semilogy(x,y,'bo')
axis([-0.1 1.1 -3 5])
hold on
hold off
plot3(x,y,z,'bo')
for k=1:10 ... end

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returns the solution x to Ax = b
returns the inverse of A
returns the reduced row echelon form of A
returns the determinant of A
returns the (operator) norm of A
returns the condition number of A
returns the larger of the number of rows and number of columns of A
returns the norm (length) of a vector x
returns the Vandermonde matrix for the points of x
returns the values of the polynomial  $a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$  at the points of x
returns the matrices Q and R in the QR factorization of A
calculates the next power of 2 of N
FFT transform of the vector f using N points (pads f with zeros if it has fewer than N elements)
returns the coefficients of the characteristic polynomial of A
returns the solutions to  $a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0$ 
returns the matrix V whose columns are normalized eigenvectors of A and the diagonal matrix D of corresponding eigenvalues
plots the points of y against the points of x using blue dots
plots the points of y against the points of x using red lines
plots y against x using a logarithmic scale for y
changes the axes of the plot to be from -0.1 to 1.1 for the x-axis and -3 to 5 for the y-axis
puts any new plots on top of the existing plot
any new plot commands replace the existing plot (this is the default)
plots the points of z against the points of x and y using blue dots
for loop taking k from 1 to 10 and performing the commands ... for each

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