

University of British Columbia
Math 307, Final

April 23, 2012
3.30-6.00pm

Name:

Student Number:

Signature:

Instructor:

Instructions:

1. No notes, books or calculators are allowed. A MATLAB/Octave formula sheet is provided.
2. Read the questions carefully and make sure you provide all the information that is asked for in the question.
3. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.
4. Answer the questions in the space provided. Continue on the back of the page if necessary.

| Question | Mark | Maximum |
|----------|------|---------|
| 1 | | 12 |
| 2 | | 12 |
| 3 | | 12 |
| 4 | | 16 |
| 5 | | 12 |
| 6 | | 16 |
| 7 | | 12 |
| 8 | | 8 |
| Total | | 100 |

1. Suppose you are given a set of N data points (x_n, y_n) , with x_n increasing, and you wish to interpolate these points with a spline function f , where $f(x)$ is given by a cubic polynomial $p_n(x)$ on each interval (x_n, x_{n+1}) , for $n = 1, \dots, N - 1$:

$$p_n(x) = a_n(x - x_n)^3 + b_n(x - x_n)^2 + c_n(x - x_n) + d_n.$$

[3]

- (a) Write down the equations required for $f(x)$ to be continuous and to pass through the data points. How many equations does this provide?

[3]

- (b) Write down the equations required for $f(x)$ to have continuous first and second derivatives. How many equations does this provide?

[2]

- (c) Now suppose $y_1 = y_N$ and you wish the spline to be *periodic*, which means adding the two conditions $f'(x_1) = f'(x_N)$ and $f''(x_1) = f''(x_N)$. Write down the two equations required for these conditions.

[4]

- (d) Write down the matrix equation to be solved for the coefficients of the polynomials in the case $N = 3$, with $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

2. Suppose that you want to solve the boundary value problem

$$f''(x) - x^2 f(x) = 1 \quad 0 \leq x \leq 1, \quad f(0) = 1, \quad f(1) = 1.$$

Let $\mathbf{F} = [f_0, f_1, \dots, f_N]^T$ be the finite difference approximation to the solution $f(x)$ at uniformly spaced points x_0, x_1, \dots, x_N .

[2]

(a) Write down finite difference approximation to $f''(x)$ at an interior point x_i .

[6]

(b) The finite difference approximation to the equations and boundary conditions can be written in the matrix form $A\mathbf{F} = \mathbf{b}$. Find the matrix A and the vector \mathbf{b} for the case $N = 4$.

[4]

(c) Suppose the boundary condition at $x = 0$ is changed to

$$f(0) = 1 - f'(0).$$

How do A and \mathbf{b} change in this case?

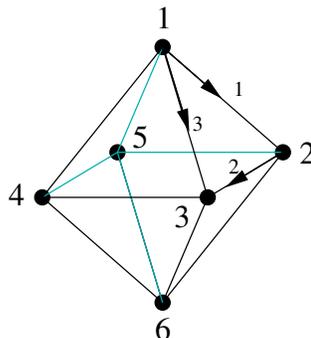
[12]

3. Suppose that A is a 3×4 matrix and

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For each of the following, either find what is asked for, or indicate that there is insufficient information to determine it: (i) the rank of A , (ii) a basis for $N(A)$, (iii) a basis for $R(A)$, (iv) a basis for $N(A^T)$, (v) a basis for $R(A^T)$.

4. Consider a resistor network arranged in the shape of an octahedron as in this diagram:



The nodes have been labeled with the large numbers and 3 of the edges (resistors) have been given orientations and labeled with small numbers. Assume all resistances are equal to 1. Let D be the incidence matrix (for some choice of labeling and arrows for the remaining edges) and let L be the Laplacian.

[2]

(a) Write down a non-zero vector in $N(D)$

[2]

(b) Write down a non-zero vector in $N(D^T)$

[3]

(c) Is it true that $N(D) = N(L)$? Give a reason.

[3]

(d) Write down the Laplacian L .

[6]

- (e) It seems reasonable to conjecture that the effective resistance between nodes 1 and 6 would remain unchanged if we removed the resistors between 2 and 3, 3 and 4, 4 and 5, 5 and 2. Explain how you could use MATLAB/Octave to test this conjecture. Assume that L has been defined in MATLAB/Octave.

[2]

5. You are given a set of 100 data points (x_n, y_n) with x_n increasing.

- (a) Suppose you wish to use Lagrange interpolation to interpolate the data points. What degree of polynomial would you use and why?

[4]

- (b) Write down the matrix equation that is satisfied by the coefficients of this polynomial. Is the numerical solution of this equation likely to be accurate, and why?

[6]

- (c) Suppose instead that you choose to make a least squares fit of the data points to a quadratic polynomial

$$f(x) = c_0x^2 + c_1x + c_2.$$

Write down the Matlab/Octave commands you would use to calculate the coefficients, plot the original data points and plot the quadratic fit (you may assume that vectors $X = [x_1, x_2, \dots, x_N]^T$ and $Y = [y_1, y_2, \dots, y_N]^T$ have already been defined).

[4]

6. (a) A continuous measurement of the temperature in Vancouver, $y(t)$, for $0 \leq t \leq T$, is decomposed as a Fourier series of the form $y(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i \omega_n t}$, where $\omega_n = n/T$.

What important property do the set of functions $\{e^{2\pi i \omega_n t}\}$ have? What is the expression for the Fourier coefficients c_n in terms of $y(t)$?

[4]

- (b) Explain the physical significance of the values $|c_n|$ and how these relate to ω_n . If t is measured in days, and the time period T encompasses several years, which values of c_n might you expect to have the largest absolute value? (*Think about what time scales you expect temperature to fluctuate over*).

[6]

- (c) Now suppose a vector \mathbf{y} contains measurements of the *daily average* temperature for 10 years. Explain how to use the Discrete Fourier Transform to produce a frequency-amplitude plot for this data. You should write the Matlab/Octave commands you would use, and include the range of frequencies 0 to 0.1 day^{-1} on your plot.

[2]

- (d) Sketch the plot that you expect to be produced by your answer to part (c).

7. Consider the MATLAB/Octave computation

```
1> A=[0 2 1;1 0 0 ;0 1 0]
```

```
A =
```

```

0   2   1
1   0   0
0   1   0
```

```
2> [S D]=eig(A)
```

```
S =
```

```

0.80902  -0.57735  0.30902
0.50000   0.57735 -0.50000
0.30902  -0.57735  0.80902
```

```
D =
```

```
Diagonal Matrix
```

```

1.61803      0      0
0 -1.00000      0
0      0 -0.61803
```

[4]

- (a) Write down a recursion relation for a sequence $x_0, x_1, x_2, x_3, \dots$ that you can analyse using this calculation. How many initial values do you have to specify to define the sequence?

[4]

- (b) If you picked initial values at random, what would you expect the large n behaviour of the resulting sequence to be? Give a reason.

[4]

- (c) Write down initial values which result in a non-zero periodic sequence where the x_n repeatedly cycle through the same values.

8. Consider the stochastic matrix

$$P = \begin{bmatrix} 0 & 1/4 & 1/3 \\ 1 & 1/2 & 1/3 \\ 0 & 1/4 & 1/3 \end{bmatrix}$$

[2]

(a) The vector $[3, 8, 3]^T$ is an eigenvector of P . What is the eigenvalue?

[3]

(b) By considering P^2 , what can you say about the other eigenvalues of P ? Give a reason.

[3]

(c) What is $\lim_{n \rightarrow \infty} P^n [1, 0, 0]^T$? Give a reason.