University of British Columbia
Math 305, Section 101 (Froese)
Final Exam, December 2014

Name (print): _____

Student ID Number: _____ **Signature:** _____

Question	Points	Score
1	7	
2	8	
3	7	
4	7	
5	8	
6	8	
7	10	
8	9	
9	10	
10	8	
11	8	
12	10	
Total:	100	

Additional Instructions:

• No notes, books or calculators are allowed.

• Read the questions carefully and make sure you provide all the information that is asked for in the question.

• Show all your work. Correct answers without explanation or accompanying work could receive no credit.

• Answer the questions in the space provided. Continue on the back of the page if necessary.

1. (7 points) What is the image of the unit circle $\{z : |z| = 1\}$ under the mapping $f(z) = z - 1/z$?

2. Let $f(z) = |z|^2$.

   (a) (3 points) For which $z = x + iy$ do the real and imaginary parts of $f(x + iy)$ have continuous partial derivatives with respect to x and y?

   (b) (3 points) For which z is $f(z)$ complex differentiable?

   (c) (2 points) For which z is $f(z)$ analytic?

3. (7 points) For which z is it true that $\text{Log}(1/z) = -\text{Log}(z)$? (Here Log denotes the principal branch of the logarithm.)

4. (7 points) What are all the values of the multivalued expression i^i?

5. Let $f(z)$ be an analytic function in some connected domain D (i.e., any two points in D can be connected with a path).

(a) (4 points) Does it follow that f has an antiderivative in D? Give a reason.

(b) (4 points) If $f'(z) = 0$ everwhere in D, does it follow that $f(z)$ is constant? Give a reason.

6. (8 points) Find $\max\limits_{z:|z|\leq 1} |z^n + i|$. Indicate what facts about analytic functions you are using in your caculation.

7. (10 points) Compute the Laurent expansion for $\dfrac{z^n}{z^2 - 1}$ for $n \in \mathbb{Z}$ valid in the region $|z| > 1$ and use it to compute $\displaystyle\oint_{|z|=5} \dfrac{z^n}{z^2 - 1}\,dz$, where the circle is traversed once in the counterclockwise direction.

8. (9 points) Explain why $\cos(1/\sqrt{z})$ is analytic in the punctured plane $\mathbb{C}\backslash\{0\}$ and compute

$$\oint_{|z|=1} \cos(1/\sqrt{z})dz,$$

where the circle is traversed once in the counterclockwise direction. Here $\sqrt{z}$ is any branch of the square root.

9. (10 points) The following integral can be computed as

$$I = \int_0^\pi \frac{\cos(2\theta)}{2+\cos(\theta)}\,d\theta = 2\pi i\Big(\operatorname{Res}\big[f(z),z_0\big] + \operatorname{Res}\big[f(z),z_1\big]\Big).$$

What are $f(z)$, z_0 and z_1?

Solution.

Since the integrand is even in θ,
$$I = \frac12\int_{-\pi}^{\pi}\frac{\cos(2\theta)}{2+\cos(\theta)}\,d\theta = \frac12\int_0^{2\pi}\frac{\cos(2\theta)}{2+\cos(\theta)}\,d\theta.$$

Let $z=e^{i\theta}$, so that $d\theta = \dfrac{dz}{iz}$, $\cos\theta = \dfrac{z+z^{-1}}{2}$, $\cos 2\theta = \dfrac{z^2+z^{-2}}{2}$.

Then
$$I = \frac{1}{2}\oint_{|z|=1}\frac{\tfrac{z^2+z^{-2}}{2}}{2+\tfrac{z+z^{-1}}{2}}\,\frac{dz}{iz} = \frac{1}{2i}\oint_{|z|=1}\frac{z^4+1}{z^2\,(z^2+4z+1)}\,dz.$$

Hence

$$f(z) = \frac{z^4+1}{2i\,z^2\,(z^2+4z+1)}.$$

The poles are $z=0$ (double) and the roots of $z^2+4z+1=0$, namely $z=-2\pm\sqrt{3}$. Only $z=-2+\sqrt{3}$ lies inside the unit circle, so

$$z_0 = 0, \qquad z_1 = -2+\sqrt{3}.$$

9. (10 points) The following integral can be computed as

$$I = \int_0^\pi \frac{\cos(2\theta)}{2+\cos(\theta)}\,d\theta = 2\pi i\Big(\operatorname{Res}\big[f(z),z_0\big] + \operatorname{Res}\big[f(z),z_1\big]\Big).$$

What are $f(z)$, z_0 and z_1?

10. (8 points) Compute

$$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 4)^2} dx.$$

11. (8 points) Compute
$$I = \int_{-\infty}^{\infty} \frac{\cos(x)}{x + 2i} dx.$$

12. (10 points) Compute
$$I = \int_0^\infty \frac{\ln(x)}{x^2 + 4} dx.$$
Include the error estimates needed to justify your calculation.