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The University of British Columbia

Sessional Exams – 2013 Term 2
Mathematics 303 Introduction to Stochastic Processes
Dr. G. Slade

Last Name: _____ First Name: _____

Student Number: _____

This exam consists of 8 questions worth 10 marks each. No aids are permitted.
Please show all work and calculations. Numerical answers need not be simplified.

Problem	total possible	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	10	
total	80	

- 1. Each candidate should be prepared to produce his library/AMS card upon request.**
- 2. Read and observe the following rules:**
- No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
- (a) Making use of any books, papers or memoranda, other than those authorized by the examiners.
 - (b) Speaking or communicating with other candidates.
 - (c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.
- 3. Smoking is not permitted during examinations.**

1. Consider the Markov chain with state space $\{0, 1, 2, 3, 4, 5\}$ and transition matrix

$$\mathbf{P} = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 3 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

- (2 marks) (a) Draw the transition diagram showing the six states with arrows indicating possible transitions and their probabilities.

- (2 marks) (b) Determine all the communicating classes of this Markov chain.

- (2 marks) (c) Determine which states are recurrent and which are transient.
- (2 marks) (d) Suppose the Markov chain starts in state 3. What is the probability it is in state 3 after two steps?
- (2 marks) (e) Suppose the Markov chain starts in state 4. Let T be the (random) time of its first return to state 4. What is the expected value of T ?

2. A Markov chain is defined as follows. There are M coins ($M \in \{1, 2, 3, \dots\}$), each showing Heads or Tails. At each step, a coin is selected uniformly at random and flipped. The state of the system is the number of Heads.

(3 marks)

- (a) Determine the transition matrix of the Markov chain.

(6 marks)

- (b) Using any method, determine the stationary distribution (your solution should prove that the distribution you find is in fact the stationary distribution).

(1 marks)

- (c) Suppose $M = 20$ and the initial state is $X_0 = 10$. How long, on average, will it take until the next time the state is 10?

3. Consider two irreducible ergodic Markov chains with the same state space $\{0, 1, \dots, N\}$, with transition matrices \mathbf{P} and \mathbf{Q} and respective stationary distributions π and ρ . We define two processes, each with initial state 0, as follows. For the process (X_n) , a fair coin is flipped and if it comes up Heads then all the following states are obtained using the transition matrix \mathbf{P} , whereas if it comes up Tails then all the following states are obtained using the transition matrix \mathbf{Q} . For the process (Y_n) , the coin is flipped at each stage, and when it is Heads then the next state is chosen using \mathbf{P} , whereas when it is Tails then the next state is chosen using \mathbf{Q} .

(6 marks) (a) Is the process (X_n) a Markov chain? Is the process (Y_n) a Markov chain? Explain your reasoning. If one or both is a Markov chain, what are the transition probabilities?

(4 marks) (b) Determine the limit $\lim_{n \rightarrow \infty} P(X_n = j)$ for each state j .

4. Consider a branching process where the number of offspring of an individual is a binomial random variable with parameters $(2, p)$, with $p \in (0, 1)$. Initially there is one individual.

(8 marks)

- (a) Calculate the probability, as a function of p , that the branching process survives forever.

(2 marks)

- (b) Suppose we observe that the initial individual has 2 offspring, and each of these then has 2 offspring of its own. What is the probability that the branching process survives forever?

5. Pizza orders arrive according to a Poisson process of rate 20 per hour. Orders are independently for a vegetarian pizza with probability $\frac{1}{4}$, and for a meat pizza with probability $\frac{3}{4}$.

(2 marks) (a) Let X denote the number of orders received between 6:45pm and 7:00pm. What kind of random variable is X (name and parameter(s))?

(2 marks) (b) Six orders arrived between 6:45pm and 7:00pm. Given this, what is the probability that fourteen orders arrive between 7:00pm and 7:45pm?

(2 marks) (c) During a particular 60 minute period, 18 orders were received. What is the probability that fewer than 4 of them were for vegetarian pizzas?

- (2 marks) (d) During a particular 60 minute period, 4 vegetarian orders were received. What is the probability that all 4 of them came during the first 30 minutes?

- (2 marks) (e) What is the probability that the first two orders after 7:00pm are for meat pizzas?

(4 marks) 6. (a) Let $\{N(t), t \geq 0\}$ be a Poisson process of rate 2. Determine $P(N(4) - N(2) = 5 | N(4) = 8)$.

(3 marks) (b) For the Poisson process of part (a), find $E(N(4) - N(2) | N(3) = 1)$.

- (3 marks) (c) Determine, with complete proof, which of the following three functions are $o(h)$ as h approaches zero and which are not: \sqrt{h} , $e^{h^2} - 1$, he^h .

7. Customers arrive at a single-server queue according to a Poisson process of rate λ . However, an arrival that finds n customers already in the system will only join the system with probability $1/(n+1)$ (in other words, such an arrival will not join the system with probability $n/(n+1)$). Service is provided at rate μ . This defines a birth death process.¹

(2 marks)

- (a) Determine all birth and death rates for this birth death process.

(7 marks)

- (b) Determine the limiting distribution of the number of customers in the system.

(1 marks)

- (c) Identify the distribution found in part (b) (give its name and any parameter(s)).

¹Recall the formulas $P_0 = (1 + \sum_{m=1}^{\infty} \frac{\lambda_{m-1} \cdots \lambda_0}{\mu_m \cdots \mu_1})^{-1}$, and $P_n = \frac{\lambda_{m-1} \cdots \lambda_0}{\mu_m \cdots \mu_1} P_0$ for $n \geq 1$.

8. A single repairperson looks after both machines 1 and 2. Each time it is repaired, machine i stays up for an exponential time with rate 1 ($i = 1, 2$). When machine i fails, it requires an exponentially distributed amount of work with rate 2 to complete its repair ($i = 1, 2$). The repairperson will always service machine 1 when it is down. For example, if machine 1 fails while machine 2 is being repaired, then the repairperson stops work on machine 2 immediately and starts on machine 1. Consider this as a continuous time Markov Chain with states:

0: both machines working,

1: machine 1 down and machine 2 working,

2: machine 1 working and machine 2 down,

3: both machines down.

(5 marks) (a) Write down the balance equations for the limiting probabilities.

(5 marks) (b) What proportion of time is machine 2 down?