

Be sure this exam has 9 pages including the cover

The University of British Columbia

Sessional Exams – 2011 Term 2  
Mathematics 303 Introduction to Stochastic Processes  
Dr. G. Slade

Last Name: \_\_\_\_\_ First Name: \_\_\_\_\_

Student Number: \_\_\_\_\_

This exam consists of 8 questions worth 10 marks each. No aids are permitted.  
Please show all work and calculations. Numerical answers need not be simplified.

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Problem	total possible	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	10	
total	80	

- 1. Each candidate should be prepared to produce his library/AMS card upon request.**
- 2. Read and observe the following rules:**
- No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
- (a) Making use of any books, papers or memoranda, other than those authorized by the examiners.
  - (b) Speaking or communicating with other candidates.
  - (c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.
- 3. Smoking is not permitted during examinations.**

1. At all times, an urn contains  $N$  balls—some white balls and some black balls. At each stage, a coin having probability  $p$  ( $0 < p < 1$ ) of landing heads is flipped. If heads appears, then a ball is chosen at random from the urn and is replaced by a white ball. If tails appears, then a ball is chosen from the urn and is replaced by a black ball. Let  $X_n$  denote the number of white balls in the urn after the  $n$ th stage. This defines a Markov chain.

(3 points) (a) Compute the transition probabilities  $P_{ij}$ .

(4 points) (b) Let  $N = 2$ . By direct calculation, find the proportion of time in each state.

(3 points) (c) For general  $N$ , guess the value of the stationary distribution  $\pi_i$  ( $i = 0, 1, \dots, N$ ).

2. Fix a number  $p$  with  $0 < p < 1$ . Consider the Markov chain on the non-negative integers  $\{0, 1, 2, \dots\}$  whose transition probabilities are given by

$$P_{n,n+1} = p, \quad P_{n,0} = 1 - p, \quad \text{for all } n \geq 0.$$

(2 points) (a) Is this Markov chain irreducible? Explain.

(2 points) (b) Is the state 0 periodic or aperiodic? Explain.

(3 points) (c) Suppose that the Markov chain is initially in state 0, and let  $T_0$  denote the time of first return to 0 (i.e.,  $T_0$  is the smallest value of  $n > 0$  such that  $X_n = 0$ , if such a value exists, and otherwise  $T_0 = \infty$ ). Determine the probability mass function of  $T_0$ .

(1 points) (d) What exactly does it mean to say that 0 is a recurrent state?

(2 points) (e) Prove that 0 a recurrent state.

- (10 points) 3. A taxi driver provides service in two zones of a city. Fares picked up in zone  $A$  will have destinations in zone  $A$  with probability 0.6 or in zone  $B$  with probability 0.4. Fares picked up in zone  $B$  will have destinations in zone  $A$  with probability 0.3 or in zone  $B$  with probability 0.7. The driver's expected profit for a trip entirely in zone  $A$  is 6; for a trip entirely in zone  $B$  is 8; and for a trip that involves both zones is 12. Find the taxi driver's expected profit per trip.

- (10 points) 4. Consider the branching process with individual offspring distribution given by  $P(\xi = n) = \frac{1}{4}(\frac{3}{4})^n$  for  $n = 0, 1, 2, \dots$ . Initially there is one individual. Compute the probability that the branching process survives forever.

5. Let  $\{N(t), t \geq 0\}$  be a Poisson process of rate  $\lambda$ . For  $s < t$ , find:

(2 points) (a)  $P(N(t) > N(s))$

(3 points) (b)  $P(N(s) = 0, N(t) = 3)$

(2 points) (c)  $E[N(t)|N(s) = 4]$

(3 points) (d)  $E[N(s)|N(t) = 4]$

6. Arrivals of the #4 bus form a Poisson process of rate 4 per hour, and arrivals of the #9 bus form an independent Poisson process of rate 6 per hour.

(2 points)

(a) What is the probability that exactly ten buses pass by in one hour?

(3 points)

(b) What is the probability that exactly one #9 bus passes by while I am waiting for a #4?

(2 points)

(c) Exactly one bus arrived during the first 10 minutes. What is the probability that it actually arrived during the first three minutes?

(3 points)

(d) When the maintenance depot goes on strike half the buses break down before they reach my stop. What, then, is the probability that I wait for 30 minutes without seeing a single bus?

7. A small barbershop, operated by a single barber, has room for at most two customers (including the one the barber is working on). Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables with mean  $\frac{1}{4}$  hour.

(6 points) (a) Set this up as a birth-death process and determine the limiting probabilities.

(2 points) (b) What is the average number of customers in the shop?

(2 points) (c) What is the proportion of potential customers that enter the shop?



8. A population of organisms consists of both male and female members. Any particular male is likely to mate with any particular female, in any time interval of length  $h$ , with probability  $\lambda h + o(h)$ . Each mating immediately produces one offspring, equally likely to be male or female. There are no deaths. Let  $N_1(t)$  and  $N_2(t)$  denote the number of males and females in the population at time  $t$ . This can be modelled as a continuous time Markov chain.

(2 points)

- (a) What is the state space?

(6 points)

- (b) What are the parameters  $v_i$  and  $P_{ij}$  (rate to leave state  $i$  and probability that the jump is to  $j$  when this happens)?

(2 points)

- (c) What do you think the limiting probabilities will be?