Marks

1. Consider the Markov chain with the transition matrix

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 2/3 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

[11] (a) Find the communicating classes. Determine their periods, and whether they are transient or recurrent.

[7] (b) If you start in state 1, what is the probability that after 4 steps you will be back in state 1?

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2. The following statements refer to a Markov chain with transition matrix P. True or false? Give brief reasons.

[4] (a) If state i is recurrent and $P_{ij}^n > 0$ for some n, then states i and j communicate.

[4] (b) Given that the process starts in a recurrent state i, the expected time to return to state i is finite.

[4] (c) There is always at least one recurrent class.

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3. At the end of each month, a certain business fires whichever member of its N-person sales staff has made the least sales during the month, and replaces him or her with someone new. Assume that each sales person is equally likely to be the one fired. Of the N people who were employed at the beginning of the year, let X_n be the number who are still there after n months (so $X_0 = N$). This constitutes a Markov chain.

[7] (a) What is the transition matrix?

[7] (b) What is the expected number of months until all of the original staff are gone?

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4. Let $N(t)$ be a Poisson process of rate λ, and S_n the time of the n'th occurrence. Find

[6] (a) $Cov(N(2), N(3))$ *Hint: independent increments*

[6] (b) $E[S_4|N(2) = 3]$

[6] (c) $E[N(2)|S_4 = 2]$

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5. Potential customers arrive at a small barbershop as a Poisson process with a rate of four per hour, but there is only room for two customers (including the one the barber is currently working on). The time to service a customer (once the barber starts working on him) has an exponential distribution with mean 20 minutes.

[6] (a) In the long run, what fraction of time is the shop empty (i.e. there no customers)?

[7] (b) Starting with a full shop, what is the expected length of time until the shop becomes empty?

[5] (c) Why do customers entering the shop **not** constitute a renewal process?

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6. People arrive at a bus stop as a Poisson process with rate 5 per hour, and begin waiting (there is room for an unlimited number to wait). However, the buses are not running today. For each individual waiting, the length of time until that individual would discover for himself or herself (if not told by anybody else) that the buses are not running is an exponential random variable with expected value 1/5 hour. Anybody who discovers that the buses are not running will announce it, and then everybody will leave immediately. Let $X(t)$ be the number of people waiting at time t.

[6] (a) Model $X(t)$ as a continuous-time Markov chain, giving the rates q_{ij}.

[5] (b) The limiting distribution for $X(t)$ as $t \to \infty$ is $\pi_j = \dfrac{1}{(e-1)(j+1)!}$. Show that this is a stationary distribution.

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[5] (c) What is the average interval between times when everybody leaves?

[4] (d) Explain briefly what the Elementary Renewal Theorem says about the answer to (c).

The End

Be sure that this examination has 8 pages including this cover

The University of British Columbia

Sessional Examinations - April 2009

Mathematics 303

Introduction to Stochastic Processes

R. Israel

Closed book examination Time: 2 $\frac{1}{2}$ hours

Name _____ **Signature** _____

Student Number _____ **Section** _____

Special Instructions:

Allowed aids: One sheet (two-sided) of notes.
Write your answers in the space provided.
Show your work.
Do not write in the boxes to the right.

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1		18
2		12
3		14
4		18
5		18
6		20
Total		100