Math 303, Section 201, April 2005

[20%] 1. The weather in Vancouver is a two-state Markov chain, and is either rainy or sunny. The probability of rain after a rainy day is 0.8, and after a sunny day is 0.3. The weather in Hong Kong is an independent two-state Markov chain, and is also either rainy or sunny. The probability of rain after a rainy day is 0.6, and after a sunny day is 0.1.

(a) Describe the weather in both cities as a 4 state Markov chain.

(b) Find the invariant distribution for this 4 state chain.

(c) What is the long run proportion of days it is raining in both Vancouver and Hong Kong?

(d) What is the long run proportion of days that the weather in both cities is sunny, but it rained in one or both cities on the previous day?

[20%] 2. A Markov chain with state space $\{0, 1, \ldots 4\}$ has the transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Find the communicating classes, and for each state find if it is transient or recurrent.

(b) What is the probability, starting in 3, that the Markov chain never visits state 1?

(c) What is the mean time, starting in 3, until the Markov chain hits state 4?

[20%] 3. Let $Z_0, Z_1, \ldots$ be a *branching process* with *offspring distribution* $p_0, p_1, p_2 \ldots$. Let $Z_0 = 1$, and let θ be the *extinction probability* for Z.

(a) Explain what is meant by the terms above in *italic*.

(b) Let $\alpha \in (0, \frac{1}{2})$, and $p_0 = \alpha$, $p_1 = 0$, $p_2 = 1 - \alpha$. Calculate the extinction probability in this case.

(c) Suppose that $\alpha = \frac{1}{4}$. What is $y = E(Z_{12}|Z_8 = 16)$? What is $P(Z_{12} = y)$?

(d) Suppose that $Z_8 = 4$. What is the probability that the population ultimately becomes extinct?

(e) Suppose that $Z_8 = 4$ and $\alpha = \frac{1}{4}$. What is the probability that all 4 of the individuals alive at time 8 have offspring alive at all future times?

[20%] 4. A bank has 2 servers S1 and S2 who have (independent) exponential service times with rates λ_1 and λ_2. Here $\lambda_2 = \theta\lambda_1$ where $\theta < 1$. Customers have to choose which service line (for S1 or S2) they join, and cannot switch lines.

(a) At time 0 customer A is being served by S1. Customers B and C enter the bank; B joins the line for S1 and C the line for S2. (So C starts being served immediately). What is the mean time until (i) B finishes being served, (ii) C finishes being served?

(b) For the situation described in (a), what is the probability that B finishes being served before C?

(c) Suppose there are n customers $A_1, A_2, \ldots, A_n$ in the line for S1 (with A_1 being served) and one customer C for S2. What is the probability that C finishes being served before A_n?

[20%] 5. Supernovae are detected somewhere in the visible universe every 3 weeks. (You may assume they occur according to a Poisson Process). They are of two types – Type I and Type II. The probability a supernova is Type II is $\frac{1}{4}$.

(a) Suppose a Type I supernova is observed today. How long does one expect to wait until the next Type I supernova?

(b) What is the probability of seeing exactly two Type II supernovae and one Type I supernova in a given period of 4 weeks?

[20%] 6. A machine is in one of three states: (A) running, (B) idle, or (C) being maintained . When it is running it runs for an exponential time with mean 1/3 hour, and it is then equally likely to become idle or be maintained. When the machine is idle, it remains idle for an exponential time with mean 1/2 hour, and is then set running. The time to maintain the machine is an exponential time with mean 1 hour. After maintenance is machine is set running with probability 2/3 and put in an idle state with probability 1/3.

(a) Describe the state of the machine as a 3-state continuous time Markov chain.

(b) What is the long run proportion of time the machine is idle?

THE END