The University of British Columbia

Final Examination - December 2009

Mathematics 263

Section 101

Closed book examination — Time: 2.5 hours

Last Name: ________________ **First:** ____________ **Signature** ________________

Student Number ________________

Special Instructions:

- Be sure that this examination has 12 pages. Write your name at the top of each page.
- You are allowed to bring into the exam one $8\frac{1}{2} \times 11$ formula sheet filled on both sides. No calculators or any other aids are allowed.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

1		20
2		15
3		20
4		20
5		10
6		15
Total		100

1. Suppose the function $T(x, y, z)$ describes the temperature at a point (x, y, z) in space, with $T(1, 1, 1) = 10$, and $\nabla T(1, 1, 1) = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$. Suppose also that the position at time t of a particle moving through space is $(\sqrt{1+t}, \cos t, e^t)$.

   (a) Compute the directional derivative of T at $(1, 1, 1)$, in the direction of the vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

   (b) At $(1, 1, 1)$, in what direction does the temperature decrease most rapidly?

(c) Compute the rate of change of temperature experienced by the particle at time $t = 0$.

(d) Write an equation for the tangent plane to the temperature level surface $T(x, y, z) = 10$ at $(1, 1, 1)$.

2. Use Lagrange multipliers to find the points on the surface $z = x^2 + 2y^2$ that are closest to the point $(0, 0, 2)$. (Hint: Minimize the distance squared rather than the distance.)

Extra space (if needed)

3. Let $\mathbf{F}(x, y, z) = \langle\, 0,\, xe^y,\, (z+1)e^z \,\rangle$.

   (a) Calculate the curl of $\mathbf{F}$.

   (b) Find a function $h(x, y, z)$ such that the vector field

   $$\mathbf{G}(x, y, z) = \langle\, h(x, y, z),\, xe^y,\, (z+1)e^z \,\rangle$$

   is conservative. Find a function $g(x, y, z)$ such that $\mathbf{G}(x, y, z) = \nabla g(x, y, z)$.

   (c) Evaluate the integral $\int_C \mathbf{G} \cdot d\mathbf{r}$, where the curve C is parametrized by $x(t) = t^2$, $y(t) = t^2$ and $z(t) = t^3$ for $0 \le t \le 1$.

   (d) Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is as in (c). (Hint: Use the results from (b) and (c).)

Extra space (if needed)

4. Let C be the closed curve oriented counterclockwise consisting of the line segment from $(0,0)$ to $(1,0)$, the line segment from $(1,0)$ to $(1,1)$ and the part of the parabola $y = x^2$ from $(1,1)$ to $(0,0)$. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = xy\,\mathbf{i} + x^2\,\mathbf{j}$ by two methods:

   (a) By calculating the line integral directly.

   (b) By using Green's Theorem.

Extra space (if needed)

5. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve in which the plane $y = 1$ intersects the sphere $x^2 + y^2 + z^2 = 5$, oriented clockwise when viewed from the positive y-axis, and

$$\mathbf{F}(x, y, z) = \left(-y^2 + e^{x^2}\right) \mathbf{i} + \ln(y^2 + y)\, \mathbf{j} + \left(x + \sqrt{z^2 + 1}\right) \mathbf{k}.$$

6. Let $\mathbf{F}(x, y, z) = \langle\, z \tan^{-1}(y^2),\ z^3 \ln(x^2 + 1),\ z \,\rangle$. Find the flux of $\mathbf{F}$ across the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upwards.

Extra space (if needed)