

Be sure that this examination has 20 pages including this cover

The University of British Columbia
Sessional Examinations - December 2016

Mathematics 257/316
Partial Differential Equations

Closed book examination

Time: $2\frac{1}{2}$ hours

Name _____ Signature _____
Student Number _____ Instructor's Name _____
Section Number _____

Special Instructions:

One $8\frac{1}{2}'' \times 11''$ formula sheet has been provided.
Notes are permitted only on the back of the formula sheet.
Show your work in the spaces provided.
Explain your solutions sufficiently to be understood.

Rules Governing Formal Examinations

1. Each candidate must be prepared to produce, upon request, a library/AMS card for identification.
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
3. No candidate shall be permitted to enter the examination room after the expiration of one half hour from the scheduled starting time, or to leave during the first half hour of the examination.
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
 - (a) Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.
 - (b) Speaking or communicating with other candidates.
 - (c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

1		20
2		20
3		20
4		20
5		20
Total		100

[20] 1.

- (a) Fourier series: Compute the Fourier sine series of $f(x)$, defined on $[0, \pi]$ by $f(x) = \cos x$.
[8 marks]

(b) Consider the following boundary value problem for Laplace's equation on half an annulus.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$

$$u(r, 0) = 0, \quad 1 < r < 2,$$

$$u(r, \pi) = 0, \quad 1 < r < 2,$$

$$u(1, \theta) = \sin 3\theta, \quad 0 < \theta < \pi.$$

$$u(2, \theta) = \cos \theta, \quad 0 < \theta < \pi.$$

Find the solution using separation of variables. [12 marks]

Question 1b continued

- [20] **2.** The motion of a string subject to a gravitational load satisfies the following initial-boundary value problem:

$$u_{tt} = a^2 u_{xx} - g, \quad 0 < x < 1, \quad t > 0 \quad (*)$$

$$u(0, t) = u(1, t) = 0,$$

$$u(x, 0) = \sin(\pi x), \quad u_t(x, 0) = 0.$$

Here g is the acceleration due to gravity and a is the wave speed. Treat both as fixed constant parameters.

- (a) Determine the static deflection of the string, which is determined by solving (*) subject to the boundary conditions, but setting $u_{tt} = 0$, i.e. it is a steady state. [8 marks]

- (b) Use the solution obtained in (a) to reduce the initial-boundary value problem to solving a homogeneous wave equation subject to homogeneous boundary conditions. Now use separation of variables to determine the solution to this boundary value problem and hence the complete solution of the entire initial-boundary value problem. [12 marks]

HINT: The following integral may be useful:

$$\int_0^1 (x^2 - x) \sin n\pi x \, dx = 2 \frac{\cos n\pi - 1}{(n\pi)^3}$$

Question 2b continued

[20] **3.** The function $f(x)$ is defined for $x \in (0, 1)$ by $f(x) = x^2 - 1/3$.
The following are 4 different Fourier series that each converge to $f(x)$ on $(0, 1)$

A: $\sum_{n=1}^{\infty} a_n \cos(n - 1/2)\pi x$, $a_n = \left(\frac{4(-1)^{n+1}}{3(n - 1/2)\pi} + \frac{4(-1)^n}{[(n - 1/2)\pi]^3} \right)$.

B: $\sum_{n=1}^{\infty} b_n \sin n\pi x$, $b_n = \left(\frac{4(-1)^{n+1} - 2}{3n\pi} + \frac{4[(-1)^n - 1]}{[n\pi]^3} \right)$.

C: $\sum_{n=1}^{\infty} c_n \cos n\pi x$, $c_n = \frac{4(-1)^n}{(n\pi)^2}$.

D: $\sum_{n=1}^{\infty} d_n \sin(n - 1/2)\pi x$, $d_n = \left(\frac{-2}{3(n - 1/2)\pi} + \frac{4(-1)^{n+1}}{[(n - 1/2)\pi]^2} - \frac{4}{[(n - 1/2)\pi]^3} \right)$.

- (a) To what value does series B converge to at $x = 1/2$? [1 mark]
- (b) To what value does series A converge to at $x = 0$? [1 mark]
- (c) To what value does series D converge to at $x = 1$? [1 mark]
- (d) To what value does series C converge to at $x = -1/2$? [1 mark]

HINT: you should not need to sum any series to evaluate these.

- (e) To what value does series A converge to at $x = -1/2$? [1 mark]
- (f) To what value does series D converge to at $x = 1$? [1 mark]
- (g) To what value does series C converge to at $x = 0$? [1 mark]
- (h) To what value does series B converge to at $x = -1/2$? [1 mark]

HINT: you should not need to sum any series to evaluate these.

Question 3 continued

- (i) Use 2 of the Fourier series given to help solve the following BVP

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 1, \quad 0 < y < 1$$

$$u(x, 0) = x^2 - 1/3, \quad 0 < x < 1,$$

$$u(x, 1) = 0, \quad 0 < x < 1,$$

$$u_x(0, y) = 0, \quad 0 < y < 1,$$

$$u_x(1, y) = y^2 - 1/3, \quad 0 < y < 1.$$

[12 marks]

Question 3i continued

[20] 4. Consider the the following IBVP

$$u_t = u_{xx} - 1, \quad 0 < x < 1, \quad t \geq 0,$$

subject to the conditions: $u(0, t) = 0$, $u(1, t) = 1$, $u(x, 0) = 0$.

- (a) Writing $u(x, t) = u_s(x) + v(x, t)$ state the problems that are satisfied by the steady state solution, $u_s(x)$, and the transient solution $v(x, t)$. [4 marks]
- (b) Find the steady state solution, $u_s(x)$. [4 marks]

- (c) Find the transient solution, $v(x, t)$ using separation of variables.
HINT: The following integral may be useful:

$$\int_0^1 (x^2 - x) \sin n\pi x \, dx = 2 \frac{\cos n\pi - 1}{(n\pi)^3}.$$

[8 marks]

Question 4c continued:

- (d) Estimate how long it takes for $|v(x, t)|$ to decay to 1% of its initial size at $x = 0.5$, i.e. when is $|v(0.5, t)| = 0.01|v(0.5, 0)|$? You may base your estimate on the 1st non-zero term in the series solution for $v(x, t)$ and may leave your estimate in the form of an expression involving \ln (or similar functions) as you don't have a calculator. [4 marks]

[20] 5. Series solutions of 2nd order differential equations.

(a) Find 2 linearly independent series solutions, $y = \sum_{k=0}^{\infty} a_k x^k$, to the DE:

$$y'' + 2xy' - y = 0.$$

What is the radius of convergence of these solutions? [7 marks]

NOTE: If you are unable to give a general formula for the coefficients, evaluate the first 3 terms in each series and give the general recurrence relationship.

Question 5a continued:

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(b) Using part (a) write down the general solution to

$$y'' + 2xy' - y = 0,$$

and find the solution that satisfies initial conditions $y(0) = 2$, $y'(0) = 0$. [4 marks]

(c) Consider the DE:

$$4xy'' + 2y' + y = 0.$$

Classify the points $0 \leq x < \infty$ as either: ordinary points, regular singular points or irregular singular points. [4 marks]

(d) Find a series solution close to $x = 0$, for the following DE

$$4xy'' + 2y' + y = 0.$$

[4 marks]

The End