Math 257-316 PDE   Formula sheet - final exam

Trigonometric and Hyperbolic Function identities

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$ $\qquad \sin^2 t + \cos^2 t = 1$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\beta\sin\alpha.$ $\qquad \sin^2 t = \frac{1}{2}(1 - \cos(2t))$

$\sinh(\alpha \pm \beta) = \sinh\alpha\cosh\beta \pm \sinh\beta\cosh\alpha$ $\qquad \cosh^2 t - \sinh^2 t = 1$

$\cosh(\alpha \pm \beta) = \cosh\alpha\cosh\beta \pm \sinh\beta\sinh\alpha.$ $\qquad \sinh^2 t = \frac{1}{2}(\cosh(2t) - 1)$

Basic linear ODE's with real coefficients

	constant coefficients	**Euler eq**
ODE	$ay'' + by' + cy = 0$	$ax^2y'' + bxy' + cy = 0$
indicial eq.	$ar^2 + br + c = 0$	$ar(r-1) + br + c = 0$
$r_1 \neq r_2$ real	$y = Ae^{r_1x} + Be^{r_2x}$	$y = Ax^{r_1} + Bx^{r_2}$
$r_1 = r_2 = r$	$y = Ae^{rx} + Bxe^{rx}$	$y = Ax^r + Bx^r\ln\lvert x\rvert$
$r = \lambda \pm i\mu$	$e^{\lambda x}[A\cos(\mu x) + B\sin(\mu x)]$	$x^\lambda[A\cos(\mu\ln\lvert x\rvert) + B\sin(\mu\ln\lvert x\rvert)]$

Series solutions for $y'' + p(x)y' + q(x)y = 0$ $(\star)$ around $x = x_0$.

Ordinary point x_0: Two linearly independent solutions of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

Regular singular point x_0: Rearrange $(\star)$ as:
$(x-x_0)^2y'' + [(x-x_0)p(x)](x-x_0)y' + [(x-x_0)^2q(x)]y = 0$
If $r_1 > r_2$ are roots of the indicial equation: $r(r-1) + br + c = 0$ where $b = \lim_{x\to x_0}(x-x_0)p(x)$ and $c = \lim_{x\to x_0}(x-x_0)^2q(x)$ then a solution of $(\star)$ is

$$y_1(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^{n+r_1} \quad \text{where } a_0 = 1.$$

The second linerly independent solution y_2 is of the form:
Case 1: If $r_1 - r_2$ is neither 0 nor a positive integer:

$$y_2(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^{n+r_2} \text{ where } \ b_0 = 1.$$

Case 2: If $r_1 - r_2 = 0$:

$$y_2(x) = y_1(x)\ln(x-x_0) + \sum_{n=1}^{\infty} b_n(x-x_0)^{n+r_2} \text{ for some } b_1, b_2 ...$$

Case 3: If $r_1 - r_2$ is a positive integer:

$$y_2(x) = ay_1(x)\ln(x-x_0) + \sum_{n=0}^{\infty} b_n(x-x_0)^{n+r_2} \text{ where } \ b_0 = 1.$$

Fourier, sine and cosine series

Let $f(x)$ be defined in $[-L, L]$then its Fourier series $Ff(x)$ is a $2L$-periodic function on $\mathbf{R}$: $Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos(\frac{n\pi x}{L}) + b_n\sin(\frac{n\pi x}{L})\right\}$
where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx$

Theorem (Pointwise convergence) If $f(x)$ and $f'(x)$ are piecewise continuous, then $Ff(x)$ converges for every x to $\frac{1}{2}[f(x-) + f(x+)]$.

Parseval's indentity

$$\frac{1}{L}\int_{-L}^{L}|f(x)|^2dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right).$$

For $f(x)$ defined in $[0, L]$, its cosine and sine series are

$$Cf(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(\frac{n\pi x}{L}), \quad a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})\,dx,$$

$$Sf(x) = \sum_{n=1}^{\infty} b_n\sin(\frac{n\pi x}{L}), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\,dx.$$

D'Alembert's solution to the wave equation

PDE: $u_{tt} = c^2u_{xx},\ -\infty < x < \infty,\ t > 0$ **IC**: $u(x,0) = f(x),\ u_t(x,0) = g(x)$.
SOLUTION: $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$

Sturm-Liouville Eigenvalue Problems

ODE: $[p(x)y']' - q(x)y + \lambda r(x)y = 0, \quad a < x < b.$
BC: $\alpha_1y(a) + \alpha_2y'(a) = 0, \quad \beta_1y(b) + \beta_2y'(b) = 0.$
Hypothesis: p, p', q, r continuous on $[a,b]$. $p(x) > 0$ and $r(x) > 0$ for $x \in [a,b]$. $\alpha_1^2 + \alpha_2^2 > 0$. $\beta_1^2 + \beta_2^2 > 0$.
Properties (1) The differential operator $Ly = [p(x)y']' - q(x)y$ is symmetric in the sense that $(f, Lg) = (Lf, g)$ for all f, g satisfying the BC, where $(f,g) = \int_a^b f(x)g(x)\,dx$. (2) All eigenvalues are real and can be ordered as $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$, and each eigenvalue admits a unique (up to a scalar factor) eigenfunction ϕ_n.
(3) **Orthogonality**: $(\phi_m, r\phi_n) = \int_a^b \phi_m(x)\phi_n(x)r(x)\,dx = 0$ if $\lambda_m \neq \lambda_n$.
(4) **Expansion**: If $f(x) : [a,b] \to \mathbf{R}$ is square integrable, then

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x),\ a < x < b\ ,\ c_n = \frac{\int_a^b f(x)\phi_n(x)r(x)\,dx}{\int_a^b \phi_n^2(x)r(x)\,dx},\ n = 1, 2, \ldots$$