

Name _____ Signature _____

UBC Student Number _____

The University of British Columbia

Final Examination – 24 April 2014

Mathematics 257/316

Partial Differential Equations/Elementary Differential Equations II

Closed book examination

Time: 150 minutes

Special Instructions:

To receive full credit, all answers must be supported with clear and correct derivations. No calculators, notes, or other aids are allowed.

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		15
2		15
3		15
4		15
5		20
6		20
Total		100

- [15] 1. (a) List all the singular points and test each one for regularity:

$$(x^2 + 2x)y'' + xy' + \frac{8}{x+2}y = 0.$$

Determine the exponents of singularity for each regular singular point you find.

Do not construct the corresponding series solutions.

- (b) Find the recurrence relation and the first four nonzero terms in a power series solution:

$$(1 - x)y'' + y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

- (c) Estimate, with a brief explanation, the radius of convergence of the series solution in part (b).

- [15] 2. A professor plans to show the class how the same function $f(x)$, $0 < x < \ell$, can be represented by four different eigenfunction series, namely,

$$(A) \quad \tilde{f}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

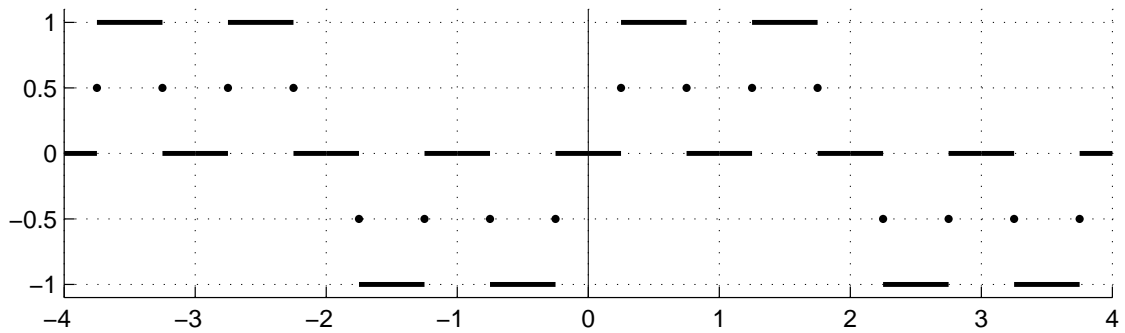
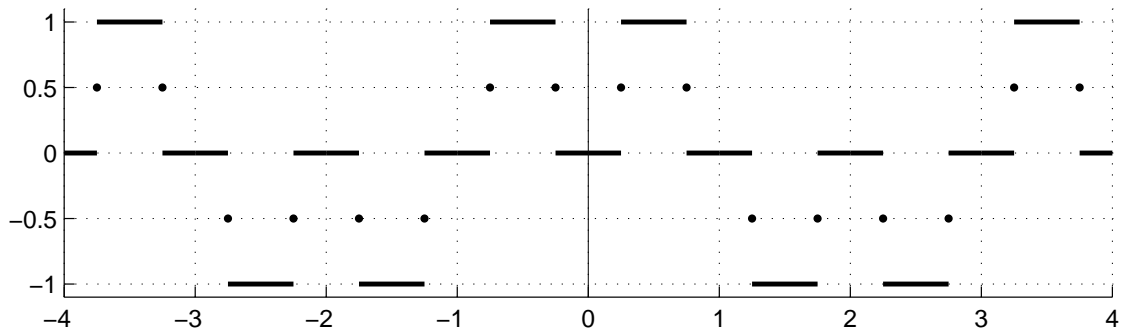
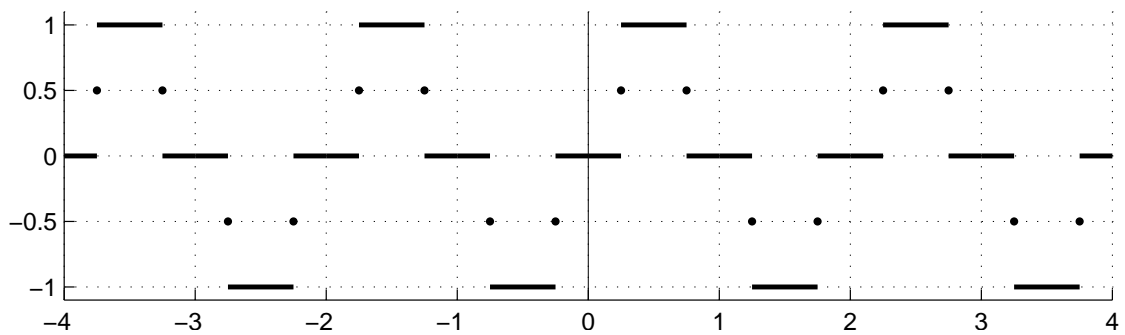
$$(B) \quad \tilde{f}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)$$

$$(C) \quad \tilde{f}(x) = \sum_{n=1}^{\infty} p_n \sin\left(\frac{(2n-1)\pi x}{2\ell}\right)$$

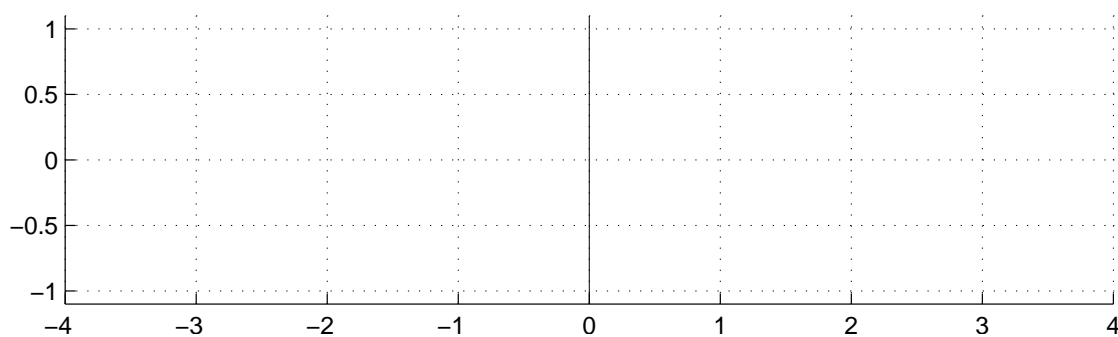
$$(D) \quad \tilde{f}(x) = \sum_{n=1}^{\infty} q_n \cos\left(\frac{(2n-1)\pi x}{2\ell}\right)$$

On the way to class, the wind blows the professor's notes away. Helpful students rescue three of the pages. Each page shows a sketch related to the professor's intended function f .

- (a) Here are the three recovered sketches. Clearly label each one with the letter (A–D) for the corresponding eigenfunction series. Write the label in the lower-left corner of each sketch.



- (b) One of the professor's sketches was lost forever. Reconstruct it by following these steps:
- Determine the value of ℓ for which expansions (A)–(D) apply.
 - Clearly identify which eigenfunction expansion (A)–(D) the lost sketch must have shown. Calculate the Fourier coefficients for that expansion. (After you solve the integrals, detailed simplification of the results is not required.)
 - Draw the lost sketch on the axes provided below.



[15] **3.** Solve for $u(x, t)$:

$$\begin{aligned} \text{(PDE)} \quad & u_{tt} = 4u_{xx} - k^2u, & 0 < x < L, \quad t > 0, \\ \text{(BC)} \quad & u_x(0, t) = 0, \quad u(L, t) = 0, & t > 0, \\ \text{(IC)} \quad & u(x, 0) = f(x), \quad u_t(x, 0) = g(x), & 0 < x < L. \end{aligned}$$

Here k is a nonnegative constant, and details about the functions f and g are not available.

Present as much detail about $u(x, t)$ as possible under these conditions—in particular, show how to use integrals involving f, g to determine all the constants in an eigenfunction-series solution.

(Extra work space for Question 3.)

- [15] 4. The steady-state temperature in a quarter-disk, $u(r, \theta)$, satisfies the following conditions:

$$\begin{aligned} \text{(PDE)} \quad & 0 = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}, & 0 < r < 1, \quad 0 < \theta < \pi/2, \\ \text{(BC)} \quad & u(r, 0) = 0, \quad u(r, \pi/2) = 0 & 0 < r < 1, \\ \text{(BC)} \quad & u_r(1, \theta) = \theta, & 0 < \theta < \pi/2. \end{aligned}$$

- (a) Using separation of variables,

- (i) derive an eigenvalue problem that will determine the eigenfunctions appearing in a series-form solution, and
- (ii) determine all the eigenfunctions for this problem.

(*Note:* If you happen to recognize the eigenvalue problem in (i), you may also remember the functions requested in (ii). In this case, just write them down; details are not required.)

- (b) Calculate an eigenfunction series solution for $u(r, \theta)$.

(Extra work space for Question 4.)

- [20] 5. (a) Determine all values for the constant B with the property that $u(x, t) \rightarrow 0$ as $t \rightarrow \infty$ for every function $u(x, t)$ satisfying

$$\begin{aligned} \text{(PDE)} \quad & u_t = \alpha^2 u_{xx} - Bu, & 0 < x < \pi, & \quad t > 0, \\ \text{(BC)} \quad & u(0, t) = 0, \quad u(\pi, t) = 0, & & \quad t > 0. \end{aligned}$$

(Here α is a positive constant, which may appear in your answer.)

- (b) Using your result from part (a), or otherwise, determine a function $v(x)$ with the property that $u(x, t) \rightarrow v(x)$ as $t \rightarrow \infty$ for every function $u(x, t)$ satisfying

$$\text{(PDE)} \quad u_t = 4u_{xx} - u + x(\pi - x), \quad 0 < x < \pi, \quad t > 0,$$

$$\text{(BC)} \quad u(0, t) = -8, \quad u(\pi, t) = 0, \quad t > 0.$$

Hint: The specified limiting behaviour is equivalent to requiring that $w(x, t) \rightarrow 0$ as $t \rightarrow \infty$, where $w(x, t) = u(x, t) - v(x)$. Justifying this property is an important part of the problem.

(c) The following boundary-value problem differs from the ones above in several significant ways:

$$\begin{array}{ll}
 \text{(PDE)} & u_t = 4u_{xx} - u + e^{-t}g(x), & 0 < x < \pi, & t > 0, \\
 \text{(BC)} & u(0, t) = 0, \quad u_x(\pi, t) = 0, & & t > 0, \\
 \text{(IC)} & u(x, 0) = 0, & & 0 < x < \pi.
 \end{array}$$

Here $g(x) = \sum_{n=1}^{\infty} p_n \sin\left(\frac{2n-1}{2}x\right)$ for coefficients p_1, p_2, \dots , that you may take as given. Find the functions $T_n(t)$ in the solution formula below:

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{2n-1}{2}x\right).$$

[20] 6. Consider the following eigenvalue problem, in which m is a real constant:

$$y'' + \lambda y = 0, \quad 0 < x < 1; \quad y(0) = 0, \quad y(1) - my'(1) = 0.$$

- (a) Determine the set of all m -values for which 0 is an eigenvalue.
- (b) Determine the set of m -values for which the given problem has at least one *negative* eigenvalue.
Hints: Define $\alpha = \sqrt{-\lambda} > 0$ to save writing. Then translate the phrase “ λ is a negative eigenvalue” into the equation “ $m\alpha = f(\alpha)$ ” for a suitable function f . Sketch and study the graph of f , paying particular attention to the sign of $f(\alpha)$, the values of $f(0)$ and $f'(0)$, and the limit of $f(\alpha)$ as $\alpha \rightarrow \infty$.

- (c) Sturm-Liouville theory guarantees that there are infinitely many eigenvalues λ satisfying $\lambda > 0$. Suppose, for this part (c) only, that $m = \frac{1}{2}$. With the aid of a suitable sketch, find an open interval of length $\pi/2$ that contains the smallest of these positive eigenvalues.

- (d) Suppose m is chosen so that the problem has exactly one negative eigenvalue, named λ_1 , and infinitely many positive eigenvalues, named $\lambda_2 < \lambda_3 < \dots$. Let $\phi_1(x), \phi_2(x), \dots$ denote the corresponding eigenfunctions. Find the constants c_n in the expansion

$$\sum_{n=1}^{\infty} c_n \phi_n(x) = g(x) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{for } 0 < x < \frac{1}{2}, \\ 0, & \text{for } \frac{1}{2} < x < 1. \end{cases}$$

Evaluate any integrals involved to arrive at algebraic expressions that show each c_n as a simple function of the corresponding eigenvalue λ_n .

Hint: The cases $n = 1$ and $n \geq 2$ require separate treatments.

(Extra work space for Question 6.)