

# Math 257/316 Final Exam

20 December 2010

Last Name: \_\_\_\_\_ First Name: \_\_\_\_\_

Student Number: \_\_\_\_\_ Signature: \_\_\_\_\_

**Instructions.** The exam lasts 2.5 hours. **No calculators or electronic devices of any kind are permitted.** A formula sheet is attached. There are **12 pages** in this test including this cover page, blank pages, and the formula sheet. Unless otherwise indicated, show all your work.

Rules governing formal examinations:

1. Each candidate must be prepared to produce, upon request, a UBC card for identification.
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
  - having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
  - speaking or communicating with other candidates; and
  - purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Problem #	Value	Grade
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. Consider the ordinary differential equation

$$2xy'' + y' - xy = 0.$$

- a) [5 marks] Classify all the points  $x \geq 0$  as ordinary points, regular singular points or irregular singular points.
- b) [15 marks] Find the first 3 non-zero terms of the series solution about  $x = 0$  satisfying

$$y(0) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} x^{1/2}y'(x) = 1.$$

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2. Consider the wave equation

$$u_{tt} = u_{xx},$$

with initial conditions

$$u(x, 0) = f(x) = \begin{cases} 2(1+x) & -1 < x < -1/2, \\ 1 & -1/2 \leq x \leq 1/2, \\ 2(1-x) & 1/2 < x < 1, \\ 0 & \text{otherwise.} \end{cases} \quad u_t(x, 0) = 0.$$

- a) [10 marks] Suppose the domain is infinite,  $-\infty < x < \infty$ . Write down d'Alembert's solution, and carefully sketch  $u(x, 0)$ ,  $u(x, 1/2)$ ,  $u(x, 1)$ .
- b) [10 marks] Suppose instead the domain is finite, with reflecting (Neumann) boundary conditions at  $x = \pm 1$ :

$$u_x(-1, t) = 0, \quad u_x(1, t) = 0,$$

Briefly describe how you would use the method of finite differences to find an approximate solution to this problem. Use the notation  $u_n^k \approx u(x_n, t_k)$  to denote the values of  $u$  on the finite difference mesh, and include how you propose to incorporate the boundary and initial conditions.

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3. [20 marks] Solve Laplace's equation for  $u(r, \theta)$  in a quarter of an annulus:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < 2, \quad 0 < \theta < \pi/2,$$

$$u_{\theta}(r, 0) = 0, \quad u_{\theta}(r, \pi/2) = 0, \quad u(1, \theta) = 0, \quad u(2, \theta) = \cos 4\theta.$$

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4. Consider the heat equation with a time-dependent boundary condition

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$$

$$u(0, t) = 0, \quad u(1, t) = e^{-t}, \quad u(x, 0) = x.$$

- a) [5 marks] By finding a suitable function  $w(x, t)$  that satisfies the boundary conditions and the initial condition, show how this problem can be transformed to the following problem for  $v(x, t)$ :

$$v_t = v_{xx} + s(x, t), \quad v(0, t) = 0, \quad v(1, t) = 0, \quad v(x, 0) = 0,$$

where  $s(x, t)$  is a source term that you should find.

- b) [15 marks] Use an eigenfunction expansion to solve for  $v$  and hence find the solution to the original problem for  $u(x, t)$ .

*Hint: You may find it helpful to know the Fourier sine series for  $x$ :*

$$x = \sum_{n=1}^{\infty} -\frac{2(-1)^n}{n\pi} \sin n\pi x \quad \text{for } 0 < x < 1.$$

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5. a) [5 marks] Find the eigenfunctions, and the equation of which the eigenvalues  $\lambda_n$  are roots, for the Sturm Liouville problem

$$X'' + \lambda^2 X = 0, \quad X'(0) = 0, \quad X'(1) = -X(1).$$

- b) [15 marks] Solve Laplace's equation in a semi-infinite strip:

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 1, \quad 0 < y < \infty,$$

$$u_x(0, y) = 0, \quad u_x(1, y) + u(1, y) = 0, \quad u(x, 0) = 1, \quad u(x, y) \rightarrow 0 \text{ as } y \rightarrow \infty,$$

giving your answer in terms of the eigenvalues,  $\lambda_n$ , and evaluating any integrals.

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