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MATH 256 - April 20, 2017
Final Exam— Duration 2.5 Hours

- Do not circle the boxes. Use **dark pen/pencils** to indicate your choice.

  – Good – Bad. Will not get identified as your chosen option.

  – Too light. – Avoid. Might or might not get identified as your chosen option.

- Do not write or mark in the shaded areas labelled *'For marker use only'* nor in the area around the dots in the corners of each page.

Please encode your student number below								Please write your details below
0	0	0	0	0	0	0	0	First name:
1	1	1	1	1	1	1	1	
2	2	2	2	2	2	2	2	Last name:
3	3	3	3	3	3	3	3	
4	4	4	4	4	4	4	4	Student number:
5	5	5	5	5	5	5	5	
6	6	6	6	6	6	6	6	
7	7	7	7	7	7	7	7	Section:
8	8	8	8	8	8	8	8	
9	9	9	9	9	9	9	9	

This exam consists of 9 doulbe-sided pages with a total of 77 points.

Instructions

- For multiple choice questions, fill in the box to the left of your chosen option. Note the box-filling guidelines above. Do not make any marks in a box that you do not want to choose (e.g. to eliminate options as you read through them).
- For all written-answer question, justifying your answers and showing your work is required for getting points. When a box is provided, place your answer in it.
- Attempt to answer all questions for partial credit.
- You may use a single double-sided hand-written 8.5" x 11" formula sheet. You may NOT use any other documents nor electronic devices of any kind (including calculators, cell phones, etc.)

If you need more space

- There are blank pages at the end. You must indicate on the page where the question is asked where to look for additional work.

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Student Conduct during Examinations

(a) Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

(b) Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

(c) No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.

(d) Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

(e) Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:

   (i) speaking or communicating with other examination candidates, unless otherwise authorized;

   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;

   (iii) purposely viewing the written papers of other examination candidates;

   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,

   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)?(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

(f) Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

(g) Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

(h) Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

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Multiple-choice questions

Instructions: For all multiple-choice questions, fill in the box to the left of the best/correct answer.

MC 1 For each of the following differential equations for the function $y(x)$, select the order of the differential equation and select whether it is linear or nonlinear.

(a) $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \cos(x)y = x^3$

(i) (1 pt) Order:

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

(ii) (1 pt) Linearity:

☐ Linear ☐ Nonlinear

(b) $2\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 4x\frac{\mathrm{d}y}{\mathrm{d}x} + x^3y = e^x$

(i) (1 pt) Order:

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

(ii) (1 pt) Linearity:

☐ Linear ☐ Nonlinear

MC 2 Consider the differential equation $y'' + ky' + 16y = \cos(at)$ where a and k are some constants.

(a) (1 pt) When $k = 0$, select the value of a for which the ODE displays resonance.
$a =$

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

(b) (1 pt) What is the smallest value of k for which the solution to the ODE does not oscillate (i.e. it does not involve trig functions)?
$k =$

☐ 0 ☐ 3 ☐ 5 ☐ 8 ☐ 13 ☐ 16

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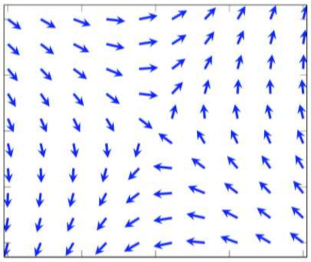
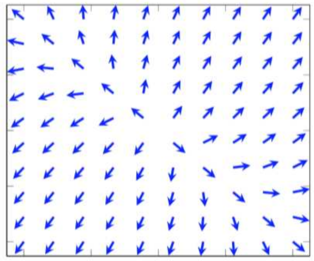
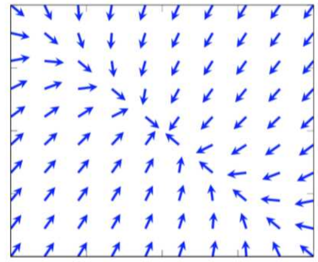
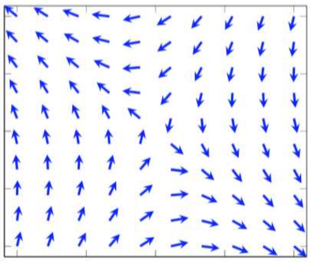
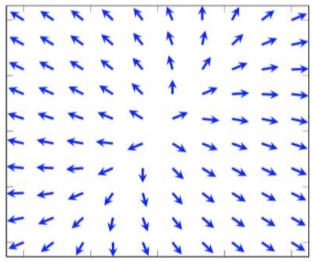
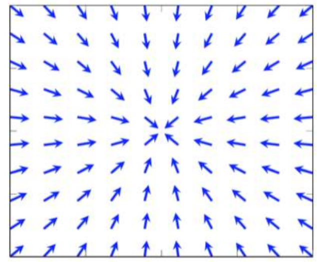
MC 3 (4 pts) The following figures show the direction fields for six different homogeneous two-dimensional systems of ordinary differential equations. Match the general solutions to the corresponding figure.

(A)

(B)

(C)


(D)

(E)

(F)


The general solution $y(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{t}$ matches the vector field in

☐ A ☐ B ☐ C ☐ D ☐ E ☐ F

The general solution $y(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-4t}$ matches the vector field in

☐ A ☐ B ☐ C ☐ D ☐ E ☐ F

The general solution $y(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$ matches the vector field in

☐ A ☐ B ☐ C ☐ D ☐ E ☐ F

The general solution $y(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$ matches the vector field in

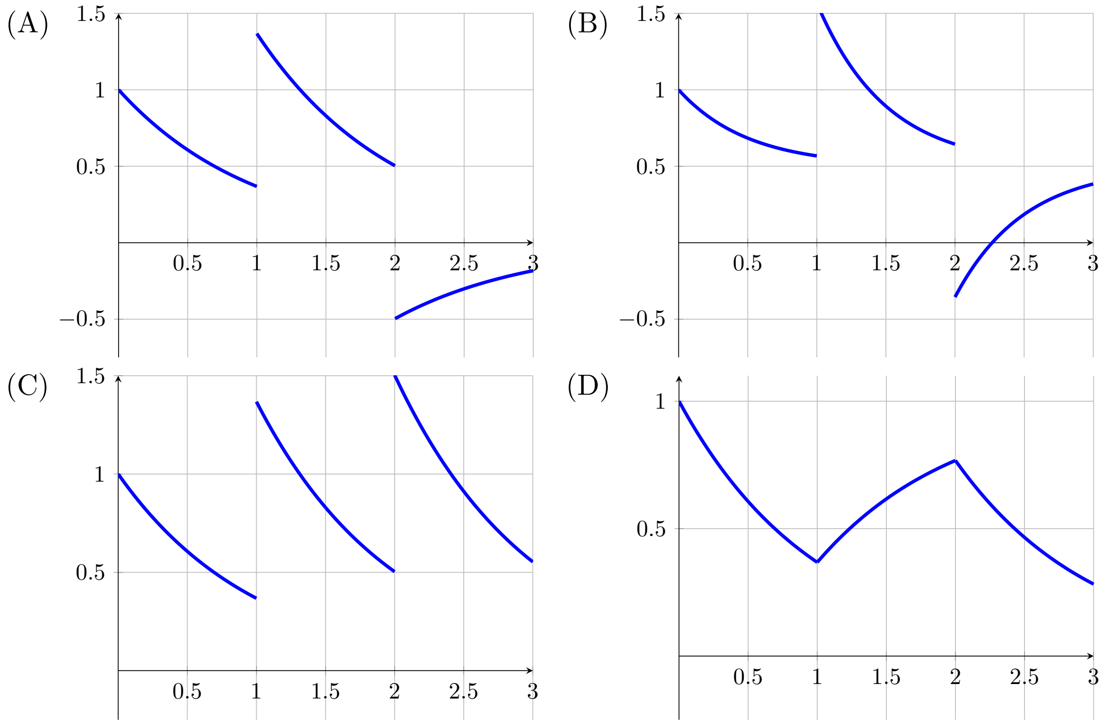
☐ A ☐ B ☐ C ☐ D ☐ E ☐ F

MC 4 (2 pts) The function $y(x) = Ae^{3t}\sin(2x) + Be^{3x}\cos(2x) + 6e^{2x}\cos(2x)$ where A and B are arbitrary constants is the general solution to which of the following second order ODEs?

- ☐ $y'' - 6y' + 13y = 24e^{3x}\sin(2x) + 6e^{3x}\cos(2x)$
- ☐ $y'' + 6y' + 13y = 24e^{3x}\sin(2x) + 6e^{3x}\cos(2x)$
- ☐ $y'' + 6y' + 13y = 24e^{2x}\sin(2x) + 6e^{2x}\cos(2x)$
- ☐ $y'' - 6y' + 13y = 24e^{2x}\sin(2x) + 6e^{2x}\cos(2x)$

MC 5 (2 pts) Consider the differential equation $y' + y = \delta(x-1) - \delta(x-2)$ for the function $y(x)$ with $y(0) = 1$, where $\delta(x)$ is the Dirac δ-function. Which of the following graphs shows the solution $y(x)$?

☐ A ☐ B ☐ C ☐ D



Answer ONLY ONE of the following two questions.

MC 6A (2 pts) Using separation of variables to solve the Helmholtz equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u,$$

with $u(x, y) = X(x)Y(y)$, which of the following equations do you need to solve?

- ☐ $X'' = -\lambda X, \quad Y'' = \lambda Y$
- ☐ $X'' = \lambda X, \quad Y'' = \lambda Y$
- ☐ $X'' = -\lambda X, \quad Y'' = (1 + \lambda)Y$
- ☐ $X'' = \lambda X, \quad Y'' = (1 + \lambda)Y$
- ☐ I answered 6B.

where λ is a constant.

MC 6B (2 pts) To find a particular solution to the equation $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ where $\det(A) = 0$ and $\mathbf{b}$ is some unknown constant vector, which y_p is the most appropriate, given that you don't know the details of A and $\mathbf{b}$?

- ☐ $\mathbf{u}t$
- ☐ $\mathbf{u}t^2 + \mathbf{v}t$
- ☐ $\mathbf{u}t + \mathbf{v}$
- ☐ $\mathbf{v}$
- ☐ I answered 6A.

where $\mathbf{u}$ and $\mathbf{v}$ are constant vectors.

Written-answer questions

Instructions: For all written-answer question, justifying your answers and showing your work is required for getting points. When a box is provided, place your answer in it.

Written-answer 1 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 — **for marking.**

Consider the following differential equation for the function $y(x)$:

$$y' = \cos(x)y$$

Find the general solution $y(x)$ in two different ways.

(a) (3 pts) Using separation of variables.

(b) (2 pts) Using an integrating factor.

Written-answer 2 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 — **for marking.**

(7 pts) The homogeneous solution to the differential equation

$$y'' + 4y = f(x)$$

is $y_h(x) = A\cos(2x) + B\sin(2x)$. For each of the following functions $f(x)$, *write down* the form of the particular solution $y_p(x)$ that will solve the differential equation. You do not need to solve for the unknown coefficients.

(a) $f(x) = x^2$

(b) $f(x) = 2x\sin(2x) + 7\cos(2x)$

(c) $f(x) = 3e^x$

(d) $f(x) = 4\sin(2x)e^x$

Written-answer 3 0 1 2 3 4 5 6 — **for marking.**

Consider the following differential equation for $y(x)$:

$$(x-1)y'' - xy' + y = 0.$$

(a) (3 pts) A solution to this ODE is $y_1(x) = e^x$. To find a second solution, set $y_2(x) = y_1(x)g(x)$. Show that $g(x)$ satisfies the following ODE

$$(x-1)g'' + (x-2)g' = 0.$$

(b) (3 pts) Given that the general solution to

$$(x-1)h' + (x-2)h = 0$$

is given by

$$h(x) = A(x-1)e^{-x},$$

and that

$$\int (x-1)e^{-x}\mathrm{d}x = -xe^{-x} + C$$

where A and C are arbitrary constants, find the general solution to the original equation above and use the boxes at the bottom, where C_1 and C_2 are arbitrary constants, to express it.

$y(x) = C_1$ $\boxed{}$ $+C_2$ $\boxed{}$

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Written-answer 4

☐0 ☐1 ☐2 ☐3 — **for marking.**

(3 pts) Consider the following differential equation for $y(x)$:

$$y'' + y = \frac{1}{\pi^2}\cos(2x) + \frac{1}{4\pi^2}\cos(4x) + \frac{1}{9\pi^2}\cos(6x) + \frac{1}{16\pi^2}\cos(8x) + \frac{1}{25\pi^2}\cos(10x).$$

The particular solution may be written in the form

$$y_p(x) = \sum_{n=1}^{5} B_n \cos(2nx).$$

Find an expression for B_n.

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Written-answer 5 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 — **for marking.**

(5 pts) Consider the following two-dimensional system of ODEs

$$\dot{\mathbf{y}} = \begin{pmatrix} 1 & -1 \\ a & 1-a \end{pmatrix} \mathbf{y},$$

where a is some constant. For what values of a is the steady state an unstable spiral? For what values of a is the steady state a stable spiral? What is the steady state when $a = 2$ (if you don't know the name, describe or sketch it)?

The steady state is a(n)...	condition on a
...unstable spiral	
...stable spiral	
	$a = 2$

Written-answer 6 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 — **for marking.**

(8 pts) Find the general solution to the following two-dimensional system of ODEs:

$$\dot{\mathbf{y}} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Written-answer 7 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ☐9 ☐10 ☐11 — **for marking.**

The Laplace transform of $y(t)$, the solution to a second order ODE, is given by

$$Y(s) = \frac{8}{s(s^2 - 4s + 8)}.$$

(a) (9 pts) Find $y(t)$. You may use formulas from your formula sheet.

(b) (2 pts) State a second order differential equation and initial conditions for $y(t)$ that would lead to the transform $Y(s)$ given above.

Written-answer 8 ☐0 ☐1 ☐2 ☐3 ☐4 — **for marking.**

(4 pts) Find the Laplace transform $F(s)$ of the following function,

$$f(t) = \begin{cases} 0 & \text{if } t < 2 \\ e^{-2t} & \text{if } t \geq 2 \end{cases},$$

by **either** calculating $F(s)$ explicitly from the integral formula **or** expressing $f(t)$ in terms of the Heaviside step function, and referring to your formula sheet. State any restrictions on the domain of $F(s)$.

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Written-answer 9 0 1 2 3 4 5 6 7 8 — **for marking.**

(8 pts) Let $f(x)$ be periodic with period two and let

$$f(x) = \begin{cases} 1 & \text{if } -1 \le t < 0, \\ 0 & \text{if } \quad 0 \le t < 1. \end{cases}$$

Calculate the coefficients of the Fourier series for $f(x)$. That is, give formulas for A_0, a_n and b_n so that $f(x) = A_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ for almost all values of x. At what values of x does the Fourier series not converge to $f(x)$?

$L =$

$A_0 =$

$a_n =$

$b_n =$

Does not converge for $x =$

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Written-answer 10 ☐0 ☐1 ☐2 ☐3 ☐4 — **for marking.**

(4 pts) The solution to the heat equation with boundary conditions $y(0,t) = 5$ and $y(2,t) = 1$ and initial condition $y(x,0) = 4 - 2x$ may be written in the form

$$y(x,t) = A_0 x + B_0 + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

(a) What is L?

(b) What are A_0 and B_0?

(c) The constants B_n can be written as

$$B_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

What is $g(x)$?

Blank page for overflow work Checked — **for marking.**

Work on this page will only be marked if you direct the marker here from the page of the original question.