



**Math 256 – Final examination**  
**University of British Columbia**  
*April 21, 2016, 3:30 pm to 6:00 pm*

Last name (print):

First name:

ID number:

*This exam is “closed book” with the exception of a single 8.5”x11” formula sheet (hand-written, one-sided, no photocopies). Calculators or other electronic aids are not allowed.*

Page	Score	Max
2		8
3		4
4		7
5		13
6		12
7		9
8		10
Total		63

**Rules governing examinations**

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (a) speaking or communicating with other candidates, unless otherwise authorized;
  - (b) purposely exposing written papers to the view of other candidates or imaging devices;
  - (c) purposely viewing the written papers of other candidates;
  - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

## Multiple choice

1. [4 pts] For each of the following equations, choose the best form for  $y_p(t)$  for the Method of Undetermined Coefficients from the list below by placing the appropriate letter (a, b, c...) in the box beside it.

$y'' + y' - 12y = 2e^{3t}$

$y'' - 3y' + 2y = 9t^3e^{3t}$

$y'' - 6y' + 9y = -4te^{3t}$

$y'' - 4y = 3t^2e^{3t}$

(a)  $y_p(t) = Ae^{3t}$

(b)  $y_p(t) = Ate^{3t}$

(c)  $y_p(t) = (At + B)e^{3t}$

(d)  $y_p(t) = At^2e^{3t}$

(e)  $y_p(t) = (At^2 + Bt + C)e^{3t}$

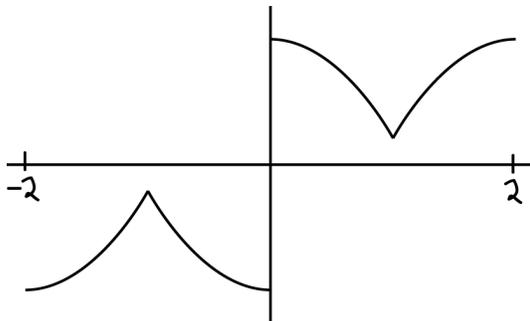
(f)  $y_p(t) = At^3e^{3t}$

(g)  $y_p(t) = (At^3 + Bt^2)e^{3t}$

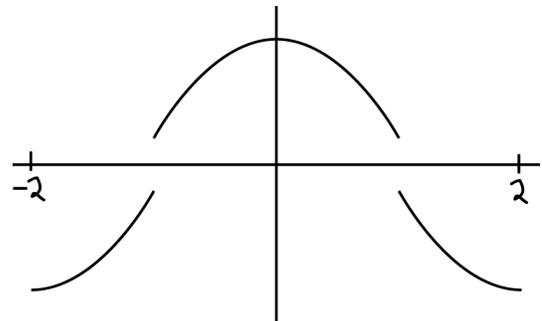
(h)  $y_p(t) = (At^3 + Bt^2 + Ct + D)e^{3t}$

2. [2 pts] Which of the following functions has a Fourier series with only sine terms of the form  $b_n \sin(n\pi x/2)$  with  $b_n = 0$  for all  $n$  even?

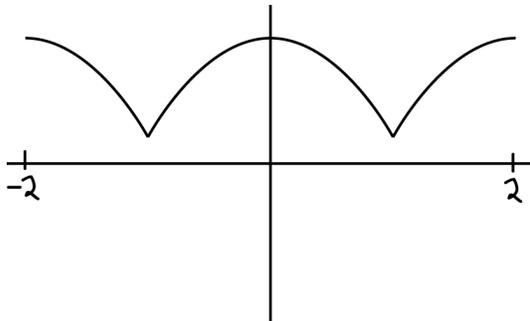
(a)



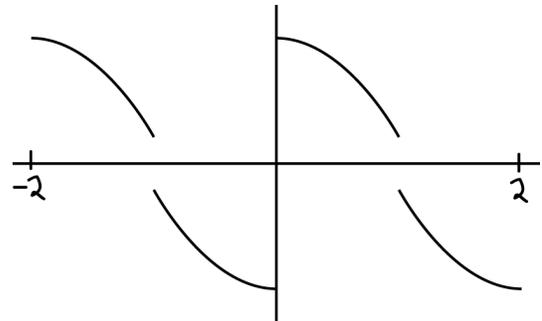
(b)



(c)



(d)



3. [2 pts] The solution to a second order linear homogeneous differential equation with constant coefficients has one non-zero solution for which  $\lim_{t \rightarrow \infty} x_1(t) = -\infty$ , another non-zero solution for which  $\lim_{t \rightarrow \infty} x_2(t) = 0$  and a third non-zero solution for which  $\lim_{t \rightarrow \infty} x_3(t) = \infty$ . Which of the following could be the differential equation in question?

(a)  $ay'' + by' + cy = 0$  where  $a > 0$ ,  $b > 0$ , and  $c > 0$ .

(b)  $ay'' + by' + cy = 0$  where  $a > 0$ ,  $b > 0$ , and  $c < 0$ .

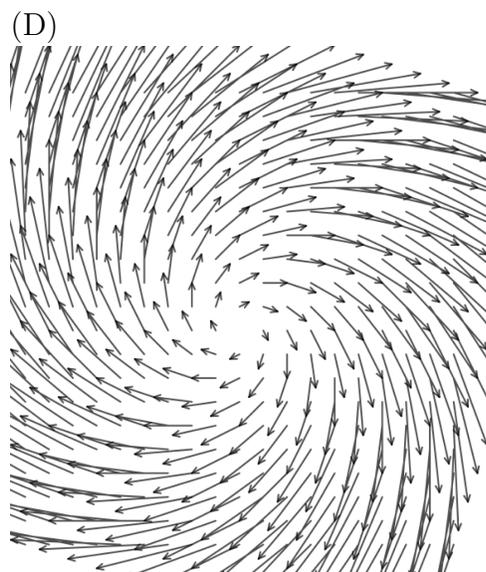
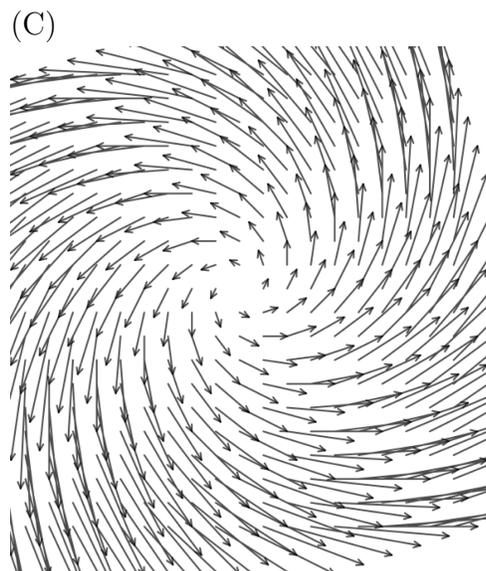
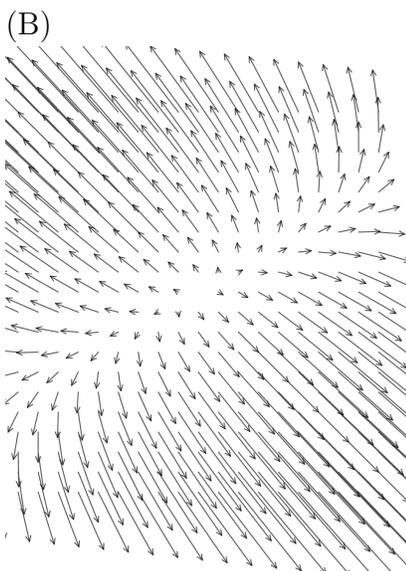
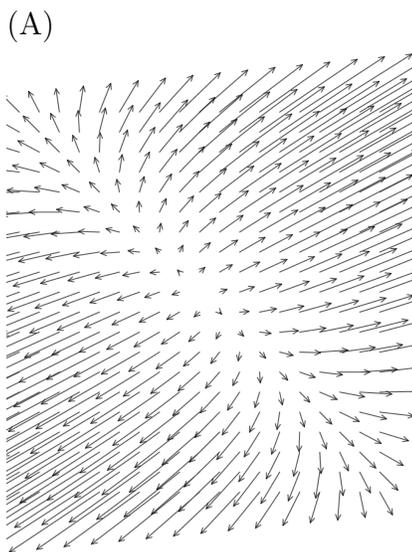
(c)  $ay'' + by' + cy = 0$  where  $a > 0$ ,  $b < 0$ , and  $c > 0$ .

(d)  $ay'' + by' + cy = 0$  where  $a < 0$ ,  $b > 0$ , and  $c < 0$ .

4. [2 pts] Which of the following pairs of functions are not independent?

- (a)  $f(t) = 3t$ ,  $g(t) = 3t - 4$
- (b)  $f(t) = (3t)^2$ ,  $g(t) = (3t - 4)^2$
- (c)  $f(t) = e^{3t}$ ,  $g(t) = e^{3t-4}$
- (d)  $f(t) = \cos(3t)$ ,  $g(t) = \cos(3t - 4)$

5. [2 pts] Choose the correct matches between the vector fields (A-D) and solutions (i-ii).



(i)  $\mathbf{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$

(ii)  $\mathbf{x}(t) = e^{2t} \left[ C_1 \begin{pmatrix} \cos(4t) \\ -\sin(4t) \end{pmatrix} + C_2 \begin{pmatrix} \sin(4t) \\ \cos(4t) \end{pmatrix} \right]$

Circle one:

- (a) (i) - (A); (ii) - (C)
- (b) (i) - (A); (ii) - (D)
- (c) (i) - (B); (ii) - (C)
- (d) (i) - (B); (ii) - (D)

## Written answers

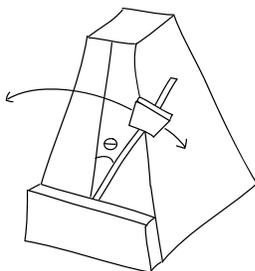
6. [7 pts] The angular deflection,  $\theta(t)$ , of a metronome (an “upside-down pendulum” used to set a beat for musicians - see sketch below) satisfies the equation  $\theta'' + 15\theta = 0$  in the absence of forcing. A truck driving by outside induces an external force on the metronome given by  $F(t) = \sum_{n=1}^{\infty} \frac{1}{n} \cos(nt)$ .

(a) What is the period of the metronome when released from a non-zero angle in the absence of forcing?

Unforced period =

(b) Find a Fourier series representation of the particular solution  $\theta(t)$  when the metronome is subject to forcing.

(c) What is the period of the largest amplitude sin/cosine function in the Fourier series of the solution?

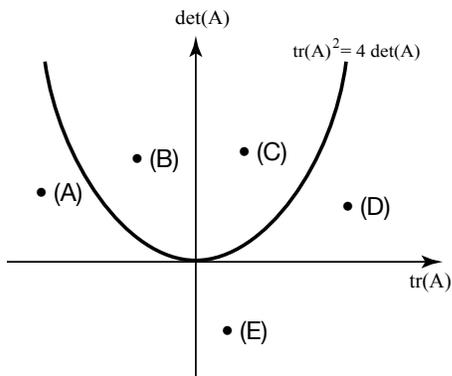


Period =

7. [7 pts] Consider the system of equations given in matrix form:

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha + 1 & 1 \\ 1 & \alpha - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Using the letters from the diagram below, list (in order) the regions of the trace/determinant plane that the system moves through as  $\alpha$  goes from  $-\infty$  to  $\infty$ .



Regions:

(b) Find the value of alpha at each transition.

8. [6 pts] Solve the Diffusion equation  $u_t = 2u_{xx}$  with boundary conditions  $\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(3, t) = 2$  and initial condition  $u(x, 0) = 2x + 5 \cos(7\pi x/3)$ .

9. [12 pts] Consider the function  $m(t) = 24 - 18(u_1(t) - u_2(t))$ .

(a) Sketch  $m(t)$ .

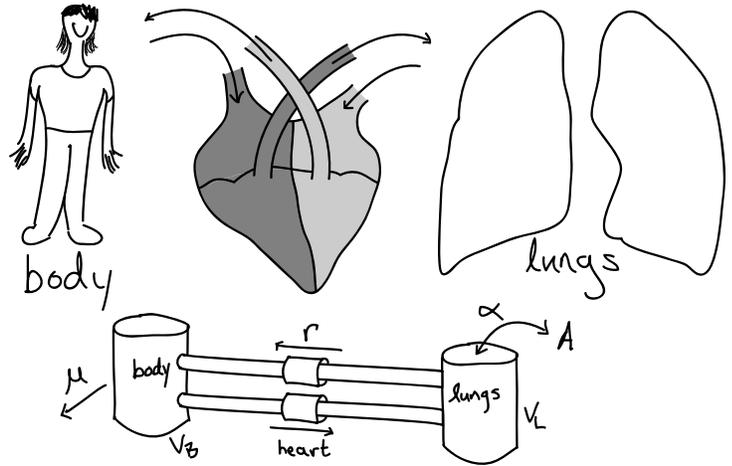
(b) Solve the differential equation  $y' + 6y = m(t)$  with initial condition  $y(0) = 0$ .

(c) Sketch the solution to the equation in (b).

10. [5 pts] Oxygenated blood (light grey) is pumped from the lungs through the heart to the body at a rate of  $r = 5\ell/\text{min}$ . Deoxygenated blood (dark grey) is pumped from the body through the heart to the lungs at the same rate.

Oxygen ( $\text{O}_2$ ) is transferred between the air and the blood in the lungs at a rate proportional to the difference between air and blood concentrations in  $\text{mol}/\ell$  with proportionality constant  $\alpha = 3\ell/\text{min}$ . The concentration of  $\text{O}_2$  in air is a constant  $A$  (in  $\text{mol}/\ell$ ).  $\text{O}_2$  is removed from the body by metabolism at a rate proportional to the number of mol of  $\text{O}_2$  in the body with a rate constant  $\mu = 2/\text{min}$ .

The volume of blood in the body is  $V_B = 5\ell$  and the volume of blood in the lungs is  $V_L = 2\ell$ . Write down a system of differential equations for the number of mol of  $\text{O}_2$  in the body,  $B(t)$ , and in the lungs,  $L(t)$ .



11. [4 pts] Calculate the Fourier series for the function  $f(x) = \delta(x - 1) - \delta(x + 1)$  when  $-2 \leq x \leq 2$  and is periodic beyond that interval.

12. [10 pts] The Laplace transform of the solution to a differential equation is given by

$$Y(s) = \frac{4}{(s+1)(s^2+2s+5)} + \frac{3}{s+1}.$$

(a) Find the solution  $y(t)$  by inverting  $Y(s)$ .

(b) Provide a differential equation and initial value(s) that have a transformed solution  $Y(s)$ .

This page is for rough work and doodles. It will not be marked.