



# Math 256 – Final examination

University of British Columbia

April 25, 2014, 12:00 pm to 2:30 pm

Name (print):

ID number:

*This exam is “closed book”. Calculators or other electronic aids are not allowed.*

Page	Score	Max
2		8
3		11
4/5		17
6		10
7		8
8		6
Total		60

### Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.

- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:

- (a) speaking or communicating with other candidates, unless otherwise authorized;

- (b) purposely exposing written papers to the view of other candidates or imaging devices;

- (c) purposely viewing the written papers of other candidates;

- (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,

- (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

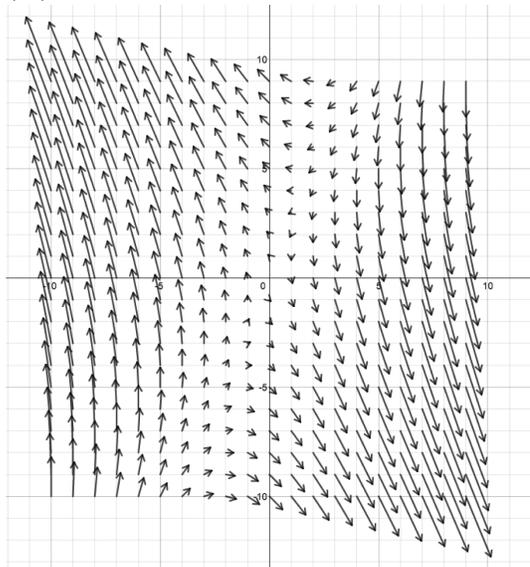
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

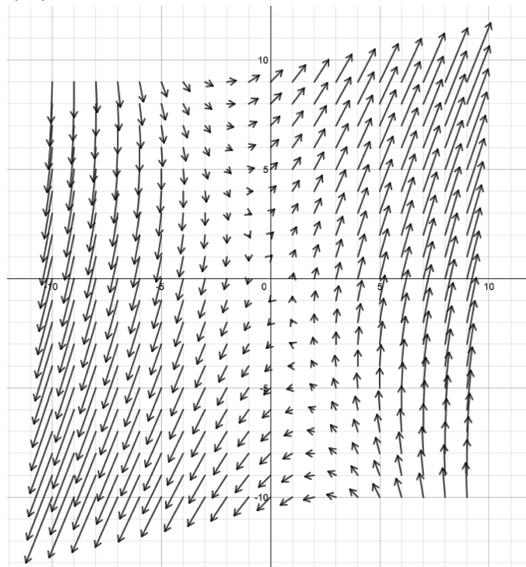
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1. [8 pts] Match the vector field to the ODE solutions below.

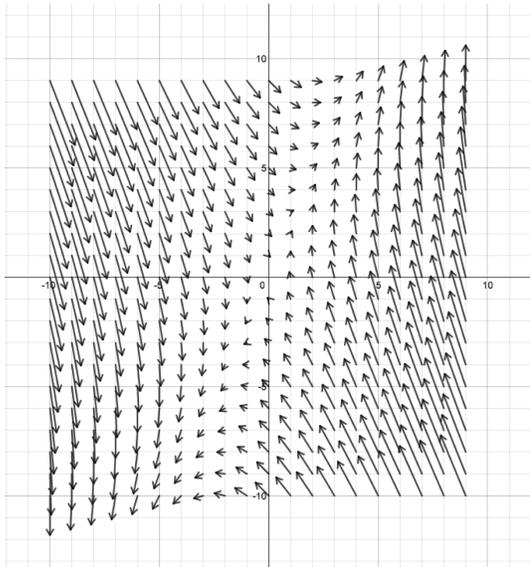
(A)



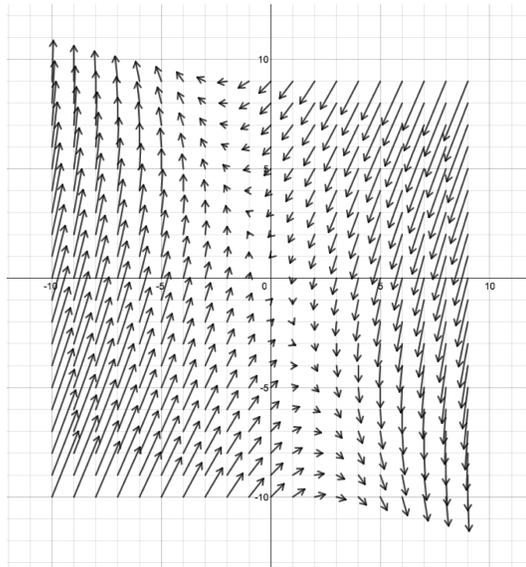
(B)



(C)



(D)



Solution	Vector field (enter A,B,C or D)
$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{6t}$	
$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-6t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t}$	
$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t}$	
$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$	

2. [6 pts] Consider the equation  $y'' + 4y' + 4y = e^{\alpha t}$ .

(a) What is the solution to the associated homogeneous problem?

(b) What is form of the particular solution (in the sense of the Method of Undetermined Coefficients)? Be sure to consider all possible cases of  $\alpha$ . Do not calculate the undetermined coefficients.

3. [5 pts] Find the solution to the initial value problem:

$$y' + y = e^{-t} \cos(t), \quad y(0) = 1.$$

4. [12 pts] Consider the Diffusion Equation  $u_t = u_{xx}$  subject to the initial condition  $u(x, 0) = x$  on the interval  $[0, 2]$ . The solution to this equation has the form

$$u(x, t) = u_s(x) + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 \pi^2 t / Q^2} \cos \frac{n \pi x}{Q} + \sum_{n=1}^{\infty} b_n e^{-n^2 \pi^2 t / Q^2} \sin \frac{n \pi x}{Q}$$

where  $Q$  is some length,

$$a_n = \frac{1}{Q} \int_{-Q}^Q g(x) \cos \frac{n \pi x}{Q} dx \quad \text{and} \quad b_n = \frac{1}{Q} \int_{-Q}^Q g(x) \sin \frac{n \pi x}{Q} dx.$$

For each of the following boundary conditions, (i) give a value of  $Q$ , (ii) determine the function  $u_s(x)$  and (iii) define the appropriate function  $g(x)$  on the interval  $[-Q, Q]$  by sketching it on the axes provided so that the expression for  $u(x, t)$  above solves the PDE, initial condition and boundary conditions. Be sure to mark the vertical axis with a number to indicate the vertical scale.

(a)

$u_x(0, t) = 0$   
 $u_x(2, t) = 0$

$Q = \underline{\hspace{2cm}}$

$u_s(x) = \underline{\hspace{4cm}}$

(b)

$u(0, t) = 8$   
 $u(2, t) = 2$

$Q = \underline{\hspace{2cm}}$

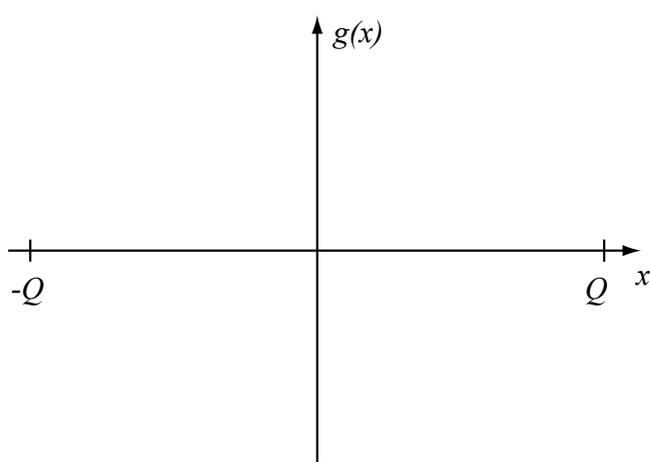
$u_s(x) = \underline{\hspace{4cm}}$

(c)

$u_x(0, t) = 0$   
 $u(2, t) = 0$

$Q = \underline{\hspace{2cm}}$

$u_s(x) = \underline{\hspace{4cm}}$

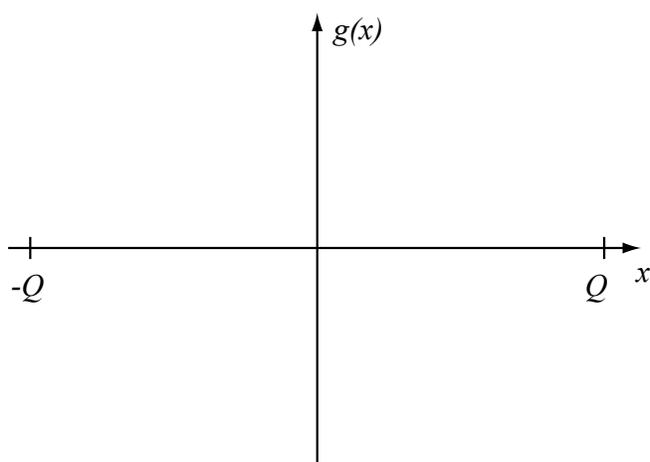


(d)

$u(0, t) = 0$   
 $u_x(2, t) = 2$

$Q = \underline{\hspace{2cm}}$

$u_s(x) = \underline{\hspace{4cm}}$



5. [5 pts] Find the inverse Laplace transform of the function

$$F(s) = \frac{s}{s^2 + 2s + 5}.$$

6. [10 pts]

(a) Find two values of  $n$  for which  $t^n$  is a solution to the equation  $t^2y'' + 6ty' + 6y = 0$ .

(b) Show that these two solutions form a fundamental set of solutions on any interval that does not include  $t = 0$ .

(c) Write down the general solution to this equation.

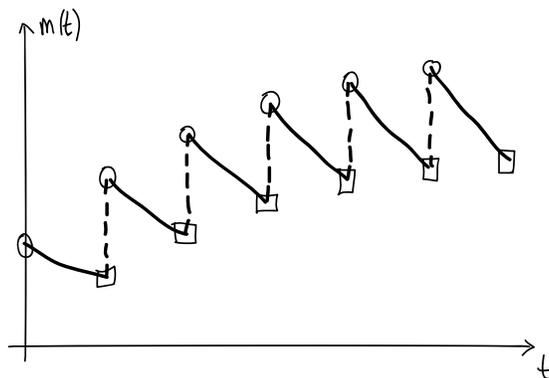
(d) Find the solution to the equation  $t^2y'' + 6ty' + 6y = 12t$  that satisfies the initial conditions  $y(1) = y'(1) = 1$ .

7. Morphine is cleared from the body at a rate proportional to the amount present with a rate constant  $k$  given in units of  $\text{hour}^{-1}$ . A patient is given an injection of  $m_0$  mg morphine immediately after surgery ( $t = 0$ ) and again every  $\tau$  hours. This continues indefinitely.

(a) [2 pts] Treating each injection as an instantaneous event, write down a differential equation to model the quantity of morphine  $m(t)$  in the patient's body as a function of time.

(b) [4 pts] Express the solution to your equation in terms of the Heaviside functions  $u_{n\tau}(t)$  where  $n = 0, 1, 2, \dots$

(c) [2 pts] The graph of the solution is shown in the figure below. Note that after a long time, the pre-injection (squares) and post-injection (circles) quantities of morphine in the patient's bloodstream asymptotically approach constant values. Calculate the asymptotic post-injection value; that is, find  $\lim_{p \rightarrow \infty} m(p\tau)$ . Hint: Depending on how you do this, the formula for a geometric series might be useful.



8. [6 pts] Consider the function  $f(x)$  to be defined on the interval  $[-2, 2]$  by

$$f(x) = \begin{cases} -1, & -2 \leq x < -1. \\ 1, & -1 \leq x < 1 \\ -1, & 1 \leq x \leq 2. \end{cases}$$

On the rest of the real line,  $f(x)$  is defined by extending it as a periodic function with period 4.

(a) Find the Fourier Series of  $f(x)$ .

(b) At what points (if any) on the real line does the Fourier Series disagree with the function  $f$ ?

This page is for rough work. It will not be marked.

**Formulae**

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$


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$$C_1 \cos(\omega t) + C_2 \sin(\omega t) = A \cos(\omega t - \phi)$$

$$A = \sqrt{C_1^2 + C_2^2}, \quad \cos(\phi) = \frac{C_1}{A}, \quad \sin(\phi) = \frac{C_2}{A}$$


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$$\omega_0^2 = \frac{k}{m}$$


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$$\omega = 2\pi f = \frac{2\pi}{T}$$


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$$\text{Critical damping: } \gamma^2 = 4mk$$


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$$r = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$


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$$W(y_1, y_2)(t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$$


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$$\sum_{n=0}^p r^n = \frac{1 - r^{p+1}}{1 - r}$$

**Laplace transforms**

$f(t)$	$F(s)$
1	$\frac{1}{s}$
$e^{at}$	$\frac{1}{s - a}$
$t^n$	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2 + a^2}$
$\cos(at)$	$\frac{s}{s^2 + a^2}$
$e^{at}f(t)$	$F(s - a)$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
$u_c(t)f(t - c)$	$e^{-sc}F(s)$
$\delta(t - c)$	$e^{-sc}$
$\int_0^t f(t - w)g(w) dw$	$F(s)G(s)$
$y'(t)$	$sY(s) - y(0)$