**Be sure that this examination has 11 pages including this cover**

**The University of British Columbia**

Sessional Examinations - April 2010

**Mathematics 256**

*Differential Equations*

Closed book examination

Time: $2\frac{1}{2}$ hours

**Name** ______________________ **Signature** ______________________

**Student Number**______________________ **Instructor's Name** ______________________

**Section Number** ______________________

## Special Instructions:

One $8\frac{1}{2}$" × 11" two sided cheat sheets are permitted.
One $8\frac{1}{2}$" × 11" one sided table of Laplace transforms is permitted.
Non-programmable calculators allowed.
Show your work in the spaces provided.
You are encouraged to explain all steps in your solutions.

### Rules Governing Formal Examinations

**1.** Each candidate must be prepared to produce, upon request, a library/AMS card for identification.

**2.** Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

**3.** No candidate shall be permitted to enter the examination room after the expiration of one half hour from the scheduled starting time, or to leave during the first half hour of the examination.

**4.** Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

**5.** Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

| | | |
|---|---|---|
| 1 | | 20 |
| 2 | | 20 |
| 3 | | 20 |
| 4 | | 20 |
| 5 | | 20 |
| Total | | 100 |

Marks

[20] **1.** The function $f(x)$ is defined for $0 \leq x \leq 2$ by $f(x) = x(2-x)$.

(a) Sketch the odd and even extensions to $f(x)$ over the range $-2 \leq x \leq 2$. Would you expect the Fourier sine series to converge to $f(x)$ faster or slower than the Fourier cosine series? Explain your reasoning in 1-2 sentences.

(b) Compute the Fourier sine series to $f(x)$.

Continued on page 3

(c) Solve the boundary value problem:

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 2, \quad 0 < y < 1.$$

subject to $u(x, 0) = 0$, $u(x, 1) = f(x)$, $u(0, y) = 0$, $u(2, y) = 0$. Here $f(x) = x(2 - x)$ as in part (a).

Continued on page 4

Marks

[20] **2.** (a) Find the general solution of the following homogeneous linear system:

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix} \mathbf{x},$$

and sketch the phase plane close to $\mathbf{x} = 0$. Is this point a saddle, a node, a spiral or a centre?

Continued on page 5

(b) Find the solution of the following initial value problem:

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -e^{-2t} \\ -6e^{-2t} \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

What is unusual about the initial conditions?

Continued on page 6

[20] **3.** (a) Solve the initial value problem:

$$(t+1)y'(t) + y(t) = -\sin t, \qquad y(0) = 1$$

(b) A forced damped oscillator has equation:

$$y''(t) + 2y'(t) + 4y(t) = \cos \omega t,$$

where the forcing frequency $\omega$ is known to lie in the range $1 \le \omega \le 5$. Find the general solution and show that as $t \to \infty$ only part of the general solution remains non-zero. This is called the steady state response of the system.

Continued on page 7

(c) Express the steady state response from part (b) in phase-amplitude form: $y_p(t) = R\cos(\omega t - \phi_0)$, showing that the amplitude $R$ depends only on $\omega^2$. What value of $\omega$ within the range $1 \leq \omega \leq 5$ will maximize the amplitude?

Continued on page 8

[20] **4.** Consider the the following initial boundary value problem

$$u_t = u_{xx} - 2, \qquad 0 < x < 2,\ t \geq 0,$$

subject to the boundary and initial conditions:

$$u(0,t) = 1, \quad u(2,t) = 1, \quad u(x,0) = 2$$

(a) Writing $u(x,t) = u_s(x) + v(x,t)$ state the problems that are satisfied by the steady state solution, $u_s(x)$, and the transient solution $v(x,t)$.

(b) Find the steady state solution, $u_s(x)$.

Continued on page 9

(c) Find the transient solution, $v(x,t)$ using separation of variables, and hence write down the full solution, $u(x,t)$.

(d) Show that

$$u(1,t) \to 0, \text{ as t} \to \infty,$$

and estimate how large $t$ must be in order for $|u(1,t)| < 0.01$.

Continued on page 10

[20] **5.** (a) Find the Laplace transform of $f(t) = t\sin t$.

(b) Find the inverse Laplace transform of

$$F(s) = \frac{5s+10}{s^2+s-6}$$

Continued on page 11

Solve the following initial value problems using Laplace transforms and describe the behaviour of $y(t)$ as $t \to \infty$:

(c)

$$y'' + 8y' + 12y = 0, \quad y(0) = 1,\ y'(0) = 0$$

.

(d)

$$y'' + 8y' + 12y = \delta(t-2) + 2u_1(t) - u_3(t), \quad y(0) = 0,\ y'(0) = 0$$

.

**The End**