Math 223, Sections 101 and 102
Final Exam
December 15, 2010
Duration: 150 minutes

Name: ______________________________ Student Number: ______________________________

Do not open this test until instructed to do so! This exam should have 17 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed. Turn off any cell phones, pagers, etc. that could make noise during the exam.

All your solutions must be written clearly and understandably. Use complete sentences and explain why your mathematical statements are relevant to the problem. If a result you are quoting requires certain hypotheses, don't forget to write that they are satisfied. You should always write enough to demonstrate that you're not just guessing the answer. Use the backs of the pages if necessary. You will find some of the questions easier than others; solve them in whatever order you like. Good luck!

Read these UBC rules governing examinations:

(i) Each candidate must be prepared to produce, upon request, a Library/AMS card for identification.

(ii) Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

(iii) No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

(iv) Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
- Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.
- Speaking or communicating with other candidates.
- Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

(v) Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Problem	Out of	Score
1	6	
2	6	
3	6	
4	8	
5	10	
6	6	

Problem	Out of	Score
7	7	
8	6	
9	9	
10	8	
11	3	
Total	75	

1. Label each of the following statements as either true or false. No justification is required; just choose the correct answer.

(a) **[1 pt]** If a system of linear equations has a solution, then it has infinitely many solutions.

True ☐ **False** ☐

(b) **[1 pt]** If a finite set of vectors is linearly dependent, then one of the vectors in the set can be written as a linear combination of the other vectors in the set.

True ☐ **False** ☐

(c) **[1 pt]** Let V be a vector space over $\mathbb{R}$ and $S \subseteq V$ be a subset. Let $W \subseteq V$ be a subspace such that $S \subseteq W$. Then $\operatorname{span}(S) \subseteq W$.

True ☐ **False** ☐

(d) **[1 pt]** Let A be a matrix. Multiplying A on the right by an elementary matrix is equivalent to performing an elementary column operation on A.

True ☐ **False** ☐

(e) **[1 pt]** If A and B are square matrices of the same size, then $\det(AB) = \det(A)\det(B)$.

True ☐ **False** ☐

(f) **[1 pt]** Let T be a linear operator on a 4-dimensional vector space. and let λ be an eigenvalue of T. If λ has algebraic multiplicity 4, then T is diagonalizable.

True ☐ **False** ☐

2. Let A be an $m \times n$ matrix with real entries. For each quantity described below, choose **Yes** if that quantity always equals $\operatorname{rank}(A)$; choose **No** if that quantity might not equal $\operatorname{rank}(A)$. No justification is required; just choose the correct answer.

(a) **[1 pt]** $\dim(R(L_A))$

Yes ☐ **No** ☐

(b) **[1 pt]** the number of nonzero columns in the reduced row echelon form of A

Yes ☐ **No** ☐

(c) **[1 pt]** the maximum number of linearly independent rows in A

Yes ☐ **No** ☐

(d) **[1 pt]** the minimum of m and n (that is, whichever number is smaller)

Yes ☐ **No** ☐

(e) **[1 pt]** $\operatorname{rank}(A^t)$

Yes ☐ **No** ☐

(f) **[1 pt]** $n - f$, where f is the number of free parameters in the set of solutions to $Ax = \underline{0}$

Yes ☐ **No** ☐

3. Let V be a finite-dimensional vector space, and let $T\colon V \to V$ be a linear operator. For each statement described below, choose **Yes** if that statement implies that T is invertible; choose **No** if that statement does not necessarily imply that T is invertible. No justification is required; just choose the correct answer.

(a) **[1 pt]** There exists a linear transformation $U\colon V \to V$ such that $UT = I_V$.

Yes ☐ **No** ☐

(b) **[1 pt]** $\operatorname{rank}(T) = \dim(V)$.

Yes ☐ **No** ☐

(c) **[1 pt]** T is onto.

Yes ☐ **No** ☐

(d) **[1 pt]** $\operatorname{nullity}(T) > 0$.

Yes ☐ **No** ☐

(e) **[1 pt]** T is the composition of two invertible linear operators on V.

Yes ☐ **No** ☐

(f) **[1 pt]** $\det([T]_\beta^\beta) \neq 0$, where β is a basis for V.

Yes ☐ **No** ☐

4. Let $\beta = \{E^{11}, E^{12}, E^{21}, E^{22}\}$ be an ordered basis for $M_{2\times 2}(\mathbb{R})$, and let $\gamma = \{1, x, x^2\}$ be the standard ordered basis for $P_2(\mathbb{R})$. Consider the linear transformation $T\colon M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (b + c + d)\, x^2 + (-a + 2b + c)\, x + (a + c + 2d).$$

(a) **[2 pts]** Write down the matrix $[T]_\beta^\gamma$.

(b) **[2 pts]** Use Gaussian elimination to transform $[T]_\beta^\gamma$ into reduced row echelon form.

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Let $\beta = \{E^{11}, E^{12}, E^{21}, E^{22}\}$ be an ordered basis for $M_{2\times 2}(\mathbb{R})$, and let $\gamma = \{1, x, x^2\}$ be the standard ordered basis for $P_2(\mathbb{R})$. Consider the linear transformation $T\colon M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (b + c + d)\,x^2 + (-a + 2b + c)\,x + (a + c + 2d).$$

(c) **[2 pts]** Find a basis for the null space $N(T)$.

(d) **[2 pts]** Find a basis for the range $R(T)$.

5. Let $\beta = \{x^2 + 2x - 1, x^2 + 2, x^2 + x\}$ and $\gamma = \{1, x, x^2\}$ be two ordered bases for $P_2(\mathbb{R})$. Suppose that $T\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ is a linear transformation such that

$$T(x^2 + 2x - 1) = x^2 + 2x - 1, \quad T(x^2 + 2) = 2x^2 + 2x, \quad \text{and } T(x^2 + x) = 10x^2 + 20.$$

(a) **[2 pts]** Write down $[T]_\beta^\beta$.

(b) **[1 pt]** Write down the change of coordinate matrix Q that changes β-coordinates into γ-coordinates (in other words, the matrix Q with the property that $Q[x]_\beta = [x]_\gamma$ for all $x \in P_2(\mathbb{R})$).

(c) **[3 pts]** With your answer Q from part (b), calculate Q^{-1}.

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Let $\beta = \{x^2 + 2x - 1, x^2 + 2, x^2 + x\}$ and $\gamma = \{1, x, x^2\}$ be two ordered bases for $P_2(\mathbb{R})$. Suppose that $T\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ is a linear transformation such that

$$T(x^2 + 2x - 1) = x^2 + 2x - 1, \quad T(x^2 + 2) = 2x^2 + 2x, \quad \text{and } T(x^2 + x) = 10x^2 + 20.$$

(d) **[3 pts]** Using your answers to parts (a), (b), and (c), calculate $[T]_\gamma^\gamma$.

(e) **[1 pt]** Using your answer to part (d), calculate $T(x^2)$.

6. Let V be a 7-dimensional subspace of $\mathbb{R}^{20}$. Let $T : V \to P(\mathbb{R})$ be a linear transformation.

(a) **[3 pts]** Prove that $R(T)$ is finite-dimensional.

(b) **[3 pts]** Suppose that $R(T) = P_4(\mathbb{R})$. Find $\mathrm{nullity}(T)$, with justification.

7.

(a) **[2 pts]** What does performing the Gram–Schmidt process on an initial set of vectors $\{w_1, \ldots, w_k\}$ accomplish? In other words, what property does the resulting set $\{v_1, \ldots, v_k\}$ have, and how is it related to the initial set $\{w_1, \ldots, w_k\}$?

(b) **[1 pt]** Suppose that the Gram–Schmidt process is performed on an initial set of vectors $\{w_1, \ldots, w_k\}$, and the resulting set $\{v_1, \ldots, v_k\}$ contains the zero vector. What does this tell us about the initial set $\{w_1, \ldots, w_k\}$?

(c) **[4 pts]** Perform the Gram–Schmidt process on the following three vectors in $\mathbb{R}^4$ (in the given order), using the standard inner product:

$$w_1 = (-2, 2, -1, 1), \quad w_2 = (6, -4, 4, -6), \quad w_3 = (1, 0, -8, 9)$$

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8. For each matrix, decide whether or not it is similar to a diagonal matrix. If not, explain why not; if so, explain why, and write down a diagonal matrix it is similar to.

(a) **[3 pts]** $A = \begin{pmatrix} 4 & 3 & -2 \\ -3 & -2 & 2 \\ 2 & 2 & -1 \end{pmatrix}$

(b) **[3 pts]** $B = \begin{pmatrix} i & -3i & 1 \\ 0 & -2i & 0 \\ -1 & 1 & i \end{pmatrix}$

9. Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

You may use, without proof, the fact that $\lambda = 1$, $\lambda = -1$, and $\lambda = 2$ are all eigenvalues of A.

(a) **[4 pts]** Find an orthonormal basis of eigenvectors of $L_A \colon \mathbb{R}^3 \to \mathbb{R}^3$, using the standard inner product.

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Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

You may use, without proof, the fact that $\lambda = 1$, $\lambda = -1$, and $\lambda = 2$ are all eigenvalues of A.

(b) **[2 pts]** Find an orthogonal matrix P and a diagonal matrix D such that $P^t AP = D$.

(c) **[1 pt]** We learned above that A has all real eigenvalues and that A is orthogonally diagonalizable (that is, there exists an orthogonal matrix P such that $P^t AP$ is diagonal). How can we tell that these things are true, just by looking at A?

(d) **[2 pts]** Find real numbers t, u, and v such that $tA^3 + uA^2 + vA = I_3$. Hint: use the Cayley–Hamilton Theorem.

10. Let V be an inner product space. Let $T\colon V \to V$ be a linear operator, and let S be a subset of V.

(a) **[2 pts]** State the definition of the adjoint linear transformation $T^*\colon V \to V$.

(b) **[3 pts]** Show that $S^\perp$ is a subspace of V.

(c) **[3 pts]** Show that $N(T) = R(T^*)^\perp$.

11. **[3 pts]** Let $A \in M_{3\times 4}(\mathbb{R})$ be a matrix satisfying $\operatorname{rank}(A) = 2$. Show that there exist matrices $B \in M_{3\times 2}(\mathbb{R})$ and $C \in M_{2\times 4}(\mathbb{R})$ such that $A = BC$.