The University of British Columbia
Final Examinations – April 25, 2010
Mathematics 221, Section 201 & 203
Instructor: D. Peterson
Time: 2.5 hours

Name: **Signature:**
Student Number: **Section Number:**

Special instructions:

1. No books or notes or electronic devices allowed.
2. Answer all questions. Each part of each question is worth 2 marks, for a total of 50 marks.
3. Give your answer in the space provided. If you need extra space, use the back of the page.
4. Show enough of your work to justify your answer. Show ALL steps.

Rules governing examinations:

1. Each candidate must be prepared to produce, upon request, a UBCcard for identification.
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
   (a) having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners; speaking or communicating with other candidates; and
   (b) purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Problem 1: Let $A = \begin{pmatrix} 1 & 2 & 3 & -2 & 4 \\ 2 & 5 & 8 & -1 & 6 \\ 1 & 4 & 7 & 5 & 2 \\ 1 & 3 & 5 & 1 & 2 \end{pmatrix}$ and let $U = \begin{pmatrix} 1 & 2 & 3 & -2 & 4 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.
Then the matrix U is an echelon form for A (you may assume this, you don't have to do the row reduction again.)

a) Find the rank of A

b) Find a basis for the column space of A

c) Find a basis of the row space of A

d) Solve the equation $A\vec{x} = 0$

e) Find a basis for the null space of A.

f) Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_5$ denote the columns of A. Express $\vec{a}_5$ as a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_4$.

g) Find the dimension of the null space of A^T.

h) Find all solutions to the system of equations:

$$
\begin{aligned}
x + 2y + 3z - 2w &= 4 \\
2x + 5y + 8z - w &= 6 \\
x + 4y + 7z + 5w &= 2 \\
x + 3y + 5z + w &= 2
\end{aligned}
$$

(Hint: what is the augmented matrix for the system?)

Problem 2: Let $u_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, u_3 = \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \end{pmatrix}$.

a) Show that the set u_1, u_2, u_3 is orthogonal.

b) Express $v = \begin{pmatrix} 3 \\ -4 \\ 1 \\ -1 \end{pmatrix}$ as a linear combination of the u_i.

c) Find a vector u_4 orthogonal to all of u_1, u_2, u_3.

Problem 3: Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 0 & -2 \\ 3 & 1 & 3 \end{pmatrix}$.

a) Write the characteristic polynomial of A

b) Find all eigenvalues of A. (You don't have to find the eigenvectors.)

Problem 4: Compute the matrix product $\begin{pmatrix} 3 & 1 & 2 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 3 & 0 & 2 \\ 5 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ -1 & 0 \\ 0 & 3 \end{pmatrix}$.

Problem 5: True or false (explain your answer): If A is a square matrix, then $\det(-A) = -\det(A)$

Problem 6: True or false (explain your answer): Suppose A is a 5×4 matrix and that b, c are vectors in $\mathbf{R}^5$. Given that $Ax = b$ has a unique solution, then $Ax = c$ has a unique solution as well.

Problem 7: Let A be a square matrix, with eigenvector v and corresponding eigenvalue λ.

a) Show that cv is also an eigenvector for A, with the same eigenvalue λ as v, for any nonzero constant c

b) Show that v is an eigenvector for the matrix $A+7I$, and find the corresponding eigenvalue.

c) If A is invertible, show that v is an eigenvector for A^{-1} and find the corresponding eigenvalue.

Problem 8: Find the line $y = a + bx$ which best fits the data points (x, y): $(0, 1), (1, 1), (1, 2)$ in the least squares sense

Problem 9: Let W be the subspace of $\mathbf{R}^3$ spanned by $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Find the vector in W which is closest to $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Problem 10: Let $A = \begin{pmatrix} 16 & -9 \\ 30 & -17 \end{pmatrix}$. Then, given that $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$ (you can assume this equation is true) find the matrix A^{10}. (There may be some powers of 2 in the answer, but you should get a single matrix.)

Problem 11: Find the determinant of the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 3 \\ 1 & 1 & 1 & 3 \end{pmatrix}$.

Problem 12: Suppose that the yearly movement of people between Alberta, BC, and Manitoba is given as follows:

- 5% of the people in Alberta move to BC, 5% move to Manitoba, and the rest remain
- 7% of the people in BC move to Alberta, 2% move to Manitoba, and the rest remain
- 6% of the people in Manitoba move to Alberta, 7% move to BC, and the the rest remain.

a) Write a matrix representing the movement of people between the 3 provinces

b) Suppose that there are 1 million people in BC. Find the populations of Manitoba and Alberta, given that the populations are in a steady state (meaning that they don't change from year to year)