The University of British Columbia

Final Examination - April 28, 2009

Mathematics 221

All Sections

Closed book examination Time: 2.5 hours

Last Name ____________________ **First** ____________ **Signature** ________________

Student Number ____________________

Special Instructions:

No notes or calculators are allowed. Answer all 12 questions on the sheets provided - use the backs of the sheets and blank sheets at the end of the test if necessary.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
Total		120

PROBLEM 1. Find all values of c such that the system of equations below is consistent. For these values of c write the general solution of the system in the parametric vector form.

$$\begin{array}{rcrcrcrcr} x_1 & & & + & 4x_3 & - & 2x_4 & = & 1 \\ -x_1 & + & x_2 & - & 7x_3 & + & 7x_4 & = & 2 \\ 2x_1 & + & 3x_2 & - & x_3 & + & cx_4 & = & 11 \end{array}$$

PROBLEM 2. Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & x & 1 \\ x & 1 & 1 & 1 \\ 1 & 1 & 1 & x \\ 1 & x & 1 & 1 \end{bmatrix}.$$

PROBLEM 3. The population $P(t)$ (in hundreds) of a colony of rabbits in year t is given in the table:

t	0	2	4	6
P	5	6	8	9

Find the equation $P(t) = a + bt$ of the least squares line that best fits the data and use it to estimate the population at time $t = 7$.

PROBLEM 4. Let $W = \text{Span}\{\vec{w}_1, \vec{w}_2\}$, where

$$\vec{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix}.$$

If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the orthogonal projection onto W, find the standard matrix of T.

PROBLEM 5. Let

$$W = \text{Span}\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Find a basis for W and a basis for the orthogonal complement $W^{\perp}$.

PROBLEM 6. If

$$x_{n+1} = 0.7x_n + 0.6y_n$$
$$y_{n+1} = 0.3x_n + 0.4y_n$$

and $x_0 = 0, y_0 = 3$, find the limiting values of x_k, y_k as $k \to \infty$.

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PROBLEM 7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection across the line $x_1 + 3x_2 = 0$, and let A be the standard the matrix of this linear transformation.

a. Find a basis for $\mathbb{R}^2$ consisting of eigenvectors of A.

b. Find the matrix A.

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PROBLEM 8. Find a formula for A^k, where

$$A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}.$$

You may leave your final answer as a product of three matrices.

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PROBLEM 9. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 3 \\ 0 & 3 & 0 & 3 \end{bmatrix}.$$

a. Find a basis for $Nul(A)$.

b. Find a basis for $Col(A)$.

c. Find the coordinate vector of $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ relative to the basis of $Col(A)$ which you found in part **b**.

d. Find the dimension of $Nul(A^T)$.

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PROBLEM 10. Find the inverses of A and AA^T, where

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

PROBLEM 11. Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

a. Find a nonzero vector $\vec{v}$ such that $A\vec{v} = 2\vec{v}$.

b. Find all eigenvalues of A.

c. Find a matrix P such that $P^{-1}AP$ is diagonal, if it exists. If such a P does not exist, explain why. (No need to find P^{-1}.)

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PROBLEM 12. Mark each statement either True or False. You do **not** have to justify your answer.

a. If A is a real $n \times n$ matrix with $A^2 = -I$, then n is even.

b. Every invertible matrix can be diagonalized.

c. If A is a 5×5 matrix such that $A + A^T = 0$, then $\det A = 0$.

d. If $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^3$, then the zero vector in $\mathbb{R}^3$ must be a linear combination of $\vec{v}$ and $\vec{w}$.

e. The sum of two eigenvectors of A is again an eigenvector of A.

f. If $T : \mathbb{R}^5 \to \mathbb{R}^4$ is a linear transformation, and if $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are vectors in $\mathbb{R}^5$ such that $T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)$ are linearly independent, then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ must be linearly independent.

g. If an $n \times n$ matrix A is not invertible, then the columns of A must be linearly dependent.

h. If a matrix A has k distinct eigenvalues, then $Rank(A) \geq k$.

i. There is no matrix A with eigenvectors $(1,1,1)^T$, $(1,0,1)^T$, and $(2,1,3)^T$ with corresponding eigenvalues $1, -1, 4$.

j. If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ is also an eigenvector of $2A$.

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