**Math 221: Final Exam**
Date: December 6, 2008

Name (please print):

Student Number:

Section number:

**Instructions**: No textbook, notes, or other aids allowed. Show all your work. If you need more space, use the back of the page. Each problem is worth 10 marks (5+5).

There are **14** pages in this exam. The last 3 pages are blank for scratch work. Please return all 14 pages.

**Problem 1a:** [5] Let $A$ denote the matrix $A = \begin{pmatrix} 3 & 0 & -2 \\ 1 & -2 & 3 \\ 4 & -2 & 1 \end{pmatrix}$. Find a basis for the column space of $A$.

**Problem 1b:** [5] Determine whether the vector $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is in the column space of the matrix $A$. above.

**Problem 2a:** [5] Let $A = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & -2 & h \end{pmatrix}$. Find all values of $h$ such that $A$ is NOT invertible.

**Problem 2b:** [5] True or false (explain your answer): if $A$ is an $n \times n$ matrix such that $\lambda = 0$ is an eigenvalue of $A$, then $A$ is not invertible.

**Problem 3:** [5] Show that the determinant of the matrix $A = \begin{pmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{pmatrix}$ is given by $(z-x)(y-x)(z-y)$.

**Problem 3b:** [5] Find the determinant of the matrix $A = \begin{pmatrix} 3 & 3 & 3 \\ 1 & 5 & 25 \\ 10 & 70 & 490 \end{pmatrix}$.

(Hint: you can use part (a) above to save some calculation.)

**Problem 4a:** [5] Let $W$ be the plane in $\mathbf{R}^4$ spanned by the vectors $v = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}$. Verify that the vector $u = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ is in the subspace $W$ and show that the vectors $u$ and $w$ form an orthogonal basis for $W$.

**Problem 4b:** [5] Find the vector in $W$ which is closest to $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

**Problem 5a:** [5] Find the eigenvalues and eigenvectors for $A = \begin{pmatrix} 5 & 0 \\ 2 & 1 \end{pmatrix}$.

**Problem 5b:** [5] Find the matrix $A^{1000}$. (Simplify your answer as much as possible.)

**Problem 6a:** [5] Suppose that each year 10% people living in Alberta move to BC, while 15% of people living in BC move to Alberta. Write a transition matrix that represents the change in population each year.

**Problem 6b:** [5] Find limiting population distribution if in the initial year, there at 100,000 people living in each province BC and Alberta.

**Problem 7a:** [5] Find the characteristic polynomial of the matrix $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 6 & 0 \end{pmatrix}$.
Give your answer in the form $-\lambda^3 + a\lambda^2 + b\lambda + c$ where $a, b, c$ are real numbers.

**Problem 7b:** [5] True or false (explain your answer): If $A$ and $B$ are similar matrices, then $A$ and $B$ have the same eigenvalues.

**Problem 8a:** [5] Suppose $A$ and $B$ are square matrices with $AB = BA$, and $B$ invertible. Then if $v$ is an eigenvector of $A$ with eigenvalue $\lambda$, show that $Bv$ is an eigenvector of $A$ with the same eigenvalue.

**Problem 8b:** [5] True or false (explain your answer): let $T$ be the transformation of $\mathbf{R}^2 \to \mathbf{R}^2$ given by $T(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$. Then $T$ is a linear transformation.

**Problem 9a:** [5] Let $B = \{b_1, b_2\}$ be a basis for $\mathbf{R}^2$ and let $T$ be the linear transformation $\mathbf{R}^2 \to \mathbf{R}^2$ such that $T(b_1) = 2b_1 + b_2$ and $T(b_2) = b_2$. Find the matrix of $T$ relative to the basis $B$.

**Problem 9b:** [5] Suppose now that $b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $b_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find the matrix of $T$ relative to the standard basis of $\mathbf{R}^2$.

**Problem 10a:** [5] True or false (explain your answer): Suppose that $v_1, \ldots, v_n$ are a basis for $\mathbf{R}^n$, and $A$ is an invertible $n \times n$ matrix. Then the vectors $Av_1, \ldots, Av_n$ are also a basis for $\mathbf{R}^n$.

**Problem 10b:** [5] True or false (explain your answer): if $A$ is a $2 \times 2$ matrix with characteristic polynomial $(\lambda - 2)^2$ then it is diagonalizable.