Final Exam

April 21, 2008, 15:30–18:00

No books. No notes. No calculators. No electronic devices of any kind.

Name (block letters) ______________________________

Student Number ______________________________

Signature ______________________________

1	2	3	4	5	6	7	8	9	10	total/65

This exam has 10 problems. The first 9 problems are common to all three sections, the last problem is section-specific.

Problem 1. (5 points)
Solve the following linear system. Your answer will depend on k.

$$\begin{aligned} x_1 + x_2 - 2x_3 + x_4 &= 1 \\ 2x_1 + 2x_2 - 3x_3 + x_4 &= 2 \\ 3x_1 + 3x_2 - 4x_3 + x_4 &= k \end{aligned}$$

Problem 2. (5 points)
Let the matrix A be

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & t \\ 0 & 3 & 1 & t \\ 1 & 0 & -1 & t \end{pmatrix}.$$

Find the inverse of A if possible. Your answer will depend on t.

Problem 3. (5 points)
Let $u = (1, 2, -1)$, $v = (-1, 0, 1)$, $w = (2, 3, -2)$ be three vectors in $\mathbb{R}^3$. Let $U = \text{span}\{u, v, w\}$ be the subspace of $\mathbb{R}^3$ spanned by u, v, w.

(a) Find the dimension of U.

(b) Find conditions on a, b, c such that the vector (a, b, c) is in U.

Problem 4. (5 points)

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y - x \end{bmatrix}$. Let $T = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ be a basis of $\mathbb{R}^2$.

(a) Find the representation of L with respect to the basis T.

(b) Let v be a vector in $\mathbb{R}^2$ such that $[L(v)]_T = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$. Find v.

Problem 5. (6 points)
Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation for which we know that

$$L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\4\\2\end{bmatrix}, \quad L\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-10\\-4\end{bmatrix}.$$

(a) What is $L\left(\begin{bmatrix}2\\0\\-3\end{bmatrix}\right)$?

(b) Find the matrix A of L with respect to the standard basis.

(c) Find $det(A)$.

Problem 6. (6 points)
Consider the homogeneous system $A\vec{x} = 0$. Suppose that the Echelon form of A is given by

$$A_1 = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is rank of A?
(b) Write down a basis for the solution space of the system $A\vec{x} = 0$.
(c) Construct an orthonormal basis for the solution space of the system $A\vec{x} = 0$.

Problem 7. (5 points)
Compute the determinant of the matrix

$$\begin{pmatrix} 0 & 3 & 0 & 2 \\ 1 & 0 & t & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 3 & 0 \end{pmatrix}$$

Problem 8. (10 points)
For each of the linear maps $\mathbb{R}^2 \to \mathbb{R}^2$, find a basis of $\mathbb{R}^2$ consisting of eigenvectors of the linear map, or explain why this is not possible.

(a) The dilation $D : \mathbb{R}^2 \to \mathbb{R}^2$ given by $D(u) = -3u$ for all $u \in \mathbb{R}^2$.

(b) The rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$, clockwise by an angle of $90°$.

(c) The reflection $S : \mathbb{R}^2 \to \mathbb{R}^2$ across the line $x = y$.

(d) The map $T = S \circ R$, (S from (c), R from (b)).

(e) The map B whose matrix is given by

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Problem 9. (9 points)

Suppose $\vec{v}_n = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$ is the state vector of a dynamical system at time n. Suppose the time dependence of the dynamical system is given by the equations

$$\begin{aligned} x_{n+1} &= x_n - y_n + z_n \\ y_{n+1} &= -x_n + 3y_n - z_n \\ z_{n+1} &= -x_n + 3y_n - z_n \end{aligned}$$

(a) Find all state vectors that do not change in time (these are vectors such that $\vec{v}_{n+1} = \vec{v}_n$, for all n).

(b) Given that $\vec{v}_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, find $\vec{v}_{100}$.

(c) Given that $\vec{v}_1 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ find all possible values for $\vec{v}_0$.

Problem 10. (9 points)
Let $q(x, y, z) = -3x^2 - 8xz + 5y^2 + 3z^2$ be a quadratic form.

(a) Find a symmetric matrix A such that $q(x, y, z) = [x\, y\, z]A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(b) The eigenvalues of A are 5 and -5. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

(c) Find a change of variables $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ and a quadratic form $q'(x', y', z')$ such that q' is equivalent to q and q' does not involve any cross-product term.