This final exam has ***9 questions*** *on* ***14 pages****, for a total of 100 marks.*

Duration: 2 hours 30 minutes

Full Name (including all middle names): ______________________________

Student-No: ______________________________

Signature: ______________________________

UBC Rules governing examinations:

1. Each candidate should be prepared to produce his/her library/AMS card upon request.

2. No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour or the *last 15 minutes* of the examination. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in the examination questions.

3. Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination, and shall be liable to disciplinary action:

   a) Making use of any books, papers or memoranda, other than those authorised by the examiners.

   b) Speaking or communicating with other candidates.

   c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness will not be received.

4. Smoking is not permitted during examinations.

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	16	8	10	7	6	11	20	10	12	100
Score:										

Please read the following points carefully before starting to write.

- Read all the questions carefully before starting to work.
- With the exception of Q1, you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- You may not leave during the first 30 minutes or final 15 minutes of the exam.

16 marks

1. Short answers question: please write the answer in complete sentences; no proofs are required.

   (a) Define carefully: a is congruent to b modulo 11 for a pair of integers a and b.

   (b) State carefully DeMorgan's laws for sets (both of them).

   (c) State (in English) the converse and contrapositive of:
   If it is not the weekend, then I get up at 7AM.

(d) State the negation of:
$\forall \epsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow \left|\frac{n}{n+1} - 2\right| < \epsilon$.

(e) Write a logical statement using only the connectives $\neg$ and $\wedge$, that is logically equivalent to $\neg(P \Rightarrow \neg Q)$.

(f) Give an example of two sets A and B, a function $f : A \to B$ and a subset C of B such that the statement

$$f(f^{-1}(C)) = C$$

is False.

(g) Give an example of two sets A and B such that A and B are both uncountable, but $|A| \neq |B|$.

8 marks 2. (a) Let $f : [0, 1] \to \mathbb{R}$ be an injection. Prove that $g : [0, \pi/2] \to \mathbb{R}$ defined by $g(x) = f(\sin x)$ is also an injection.

(b) Assume $f_1 : A \to B$ and $f_2 : B \to C$. If $f_2 \circ f_1$ is injective, prove that f_1 is injective.

10 marks 3. Let $f : [0, 1] \to \mathbb{R}$ be the function defined by $f(x) = 2x^2 + 4$.

(a) Find a set $B \subset \mathbb{R}$ such that $f : [0, 1] \to B$ is a bijection (As always, you must provide a proof.)

(b) Find a formula for the inverse function $f^{-1}(y)$ (this includes specifying its domain).

7 marks 4. Let $a_1 = 1$, $a_2 = 4$ and $a_n = 5a_{n-1} - 4a_{n-2}$ for $n \geq 3$. Prove that for all natural numbers n, $a_n = 4^{n-1}$.

6 marks 5. Find the limit as $n \to \infty$ of

$$a_n = \frac{3n^2 - 3}{2n^2 + n + 2}.$$

(As always, make sure you provide a proof.)

11 marks 6. Let $p_1, p_2, \ldots, p_n, \ldots$ be all the positive prime numbers listed in increasing order (so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \ldots$).

For $k \in \mathbb{N}$, let

$$A_k = \{a \in \mathbb{N} \mid a \geq 2 \text{ and } p_k \text{ does not divide } a\}$$

and for $n \in \mathbb{N}$, define

$$B_n = \bigcap_{k=1}^{n} A_k.$$

(a) Find the smallest element of the set B_4.

(b) Prove that for every $n \in \mathbb{N}$, the set B_n is infinite.

(c) Find $\bigcap_{k=1}^{\infty} A_k$.

20 marks 7. (a) Prove that $\sqrt{22}$ is irrational.

(b) Let $\mathbb{Z}(\sqrt{22})$ be the set of numbers of the form $a + b\sqrt{22}$, where a and b are integers.

(i) Prove that $\mathbb{Z}(\sqrt{22}) \cap \mathbb{Q} = \mathbb{Z}$.

(ii) Prove that if $x \in \mathbb{Z}(\sqrt{22})$, then for all natural numbers n, $x^n \in \mathbb{Z}(\sqrt{22})$.

(iii) Prove that $\mathbb{Z}(\sqrt{22})$ is denumerable.

10 marks 8. (a) Prove that if $|A| = |B|$, then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

(b) Let A be the set of integers divisible by 2013. Prove that $\mathcal{P}(A)$ is uncountable.

12 marks 9. (a) State the definition of the least upper bound of a set $S \subset \mathbb{R}$.

(b) Find (with proof) the least upper bound of the set $S = \{2 - \frac{3}{n^2} \mid n \in \mathbb{Z}\}$.

(c) Prove that there is no non-empty set of real numbers, A, such that the set of upper bounds of A equals $(1, \infty)$.