*This final exam has **8 questions** on **13 pages**, for a total of 100 marks.*

Duration: 2 hours 30 minutes

Full Name (including all middle names): ______________________________

Student-No: ______________________________

Signature: ______________________________

UBC Rules governing examinations:

1. Each candidate should be prepared to produce his/her library/AMS card upon request.
2. No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour or the *last 15 minutes* of the examination. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in the examination questions.
3. Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination, and shall be liable to disciplinary action:

   a) Making use of any books, papers or memoranda, other than those authorised by the examiners.

   b) Speaking or communicating with other candidates.

   c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness will not be received.
4. Smoking is not permitted during examinations.

Question:	1	2	3	4	5	6	7	8	Total
Points:	20	12	8	10	10	20	10	10	100
Score:									

Please read the following points carefully before starting to write.

- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked — except where specifically stated.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.).
- You may not leave during the first 30 minutes or final 15 minutes of the exam.
- Read all the questions carefully before starting to work.
- Continue on the back of the previous page if you run out of space.

20 marks

1. Please give precise answers to the following:

   (a) (3 marks) State De Morgan's laws (in the context of sets).

   (b) (3 marks) Define the sets $X = \bigcap_{\alpha \in I} S_\alpha$ and $Y = \bigcup_{\alpha \in I} S_\alpha$.

   (c) (3 marks) Define what it means for two sets A and B to have the same cardinality.

(d) (3 marks) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$ be two permutations of $\{1, 2, 3, 4\}$. Find the composition $\alpha \circ \beta$.

(e) (3 marks) Let A, B be non-empty sets, and let $f : A \to B$ be a function. When does f have an inverse function f^{-1}? Define f^{-1}.

(f) (5 marks) State the strong principle of mathematical induction.

12 marks 2. For $n \in \mathbb{N}$, let $P(n)$ be the statement "n is divisible by 4", and let $Q(n)$ be the statement "$n^2 + 1$ is divisible by 3".

(a) (6 marks) Consider the statement

$$\forall n \in \mathbb{N},\ P(n) \Rightarrow Q(n)$$

Is this statement true or false? Prove your answer.

(b) (6 marks) Write out in words the statement "$P(3) \Rightarrow Q(4)$", its converse and its contrapositive.

8 marks 3. For statements P, Q, and R, prove that the statement

$$[(P \Rightarrow Q) \Rightarrow R] \vee [\sim P \vee Q]$$

is a tautology.

10 marks

4. Let $f : A \to B$ be a function. If $C \subseteq A$, define

$$f(C) = \{f(x) : \ x \in C\}.$$

Prove that if f is injective, then $f(C \cap D) = f(C) \cap f(D)$ for all sets $C, D \subseteq A$.

10 marks 5. (a) (2 marks) State what it means for a set A to be denumerable.

(b) (8 marks) Let A be a denumerable set. Prove that there exists a proper subset B of A such that B is denumerable.

20 marks 6. Use mathematical induction to prove the following statements:

(a) (10 marks) Prove that $1 + 3 + 5 + \cdots + (2n - 1)$ is a perfect square for all $n \in \mathbb{N}$. (An integer m is a perfect square if $m = k^2$ for some integer k.)

(b) (10 marks) For all $n \in \mathbb{N}$, $12^{2n-1} + 11^{n+1}$ is a multiple of 133.

10 marks 7. In this question, correct answers without proofs are sufficient.

(a) (4 marks) Find a function $f : \mathbb{Z} \to \mathbb{Z}$ which is injective but not surjective.

(b) (6 marks) Find a function $g : \mathbb{Z} \to \mathbb{Z}$ which is surjective but not injective.

10 marks 8. Prove that if p is a prime number and $p > 4$, then $p^2 \equiv 1 \pmod{6}$. (Hint: Think about the possible remainders when p is divided by 6.)

This page has been left blank for your workings and solutions.